Spring 2011 Qualifying Exam

Part II

Mathematical tables are provided. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 \times 10^{-34}$ Js, $\hbar = 1.054571628 \times 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 \times 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 \times 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 \times 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 \times 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 \times 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 \times 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 \times 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 \times 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e\, c) = 2.42631 \times 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 \times 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 \times 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 \times 10^{-3}$ m K

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:
A particle of mass $m = 1$ oscillates without friction attached to a spring with $k = 4$. The motion of the particle is driven by the external force $F(t) = 3\,t\cos(t)$. Find the equation of motion and solve it. Discuss the physical meaning of the solution.

Problem 2:
Let A and B be two observables (Hermitian operators). In any state of the system
$\Delta A \Delta B \geq (1/2)|\langle i[A,B]\rangle|$.
(a) Prove this generalized uncertainty principle.
[Hint: Let $|\psi\rangle$ be any state vector and let $A_1 = A - \langle A\rangle I$ and $B_1 = B - \langle B\rangle I$.
Let $|\phi\rangle = A_1|\psi\rangle + ixB_1|\psi\rangle$ with x an arbitrary real number. Use $\langle\phi|\phi\rangle \geq 0$.]
Now consider a single particle in an eigenstate of L^2 with wave function $\Psi(\mathbf{r},t)$.
(b) Calculate the commutators $[\sin\phi, L_z]$ and $[\cos\phi, L_z]$, where ϕ is the azimuthal angle.
(c) Use these commutation relations and the result from part (a) to obtain uncertainty relations between $\sin\phi$, L_z and $\cos\phi$, L_z.

Note: You can complete parts (b) and (c) without completing part (a).

Problem 3:
A disk of radius a carries a non-uniform surface charge density given by $\sigma = \sigma_0 r^2/a^2$, where σ_0 is a constant.
(a) Find the electrostatic potential at an arbitrary point on the disk axis, a distance z from the disk center and express the result in terms of the total charge Q.
(b) Calculate the electric field on the disk axis and express the result in terms of the total charge Q.
(c) Show that the field reduces to an expected form for $z \gg a$.
(d) To first order in ρ, find an expression for the radial component of $\mathbf{E}(\rho, \phi, z)$ at a distance $\rho \ll a$ away from the z-axis and evaluate it for $z \gg a$.

Problem 4:

Consider a spinless particle in a three-dimensional potential, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2}r^2.$$

(a) Write down the ground state energy eigenfunction and eigenvalue. (You do not have to derive it, just write it down.)

(b) Find the expectation value of r, <r>.

(c) Find the energy eigenvalues and determine the degeneracies of the four lowest energy eigenstates.

(d) Now assume that 5 identical particles are in this potential. What is the ground state energy of this system if these particles have

(i) spin ½,

(ii) spin 1,

(iii) spin 3/2?

Assume that these particles do not interact with each other.

Problem 5:

A quantity of an ideal monatomic gas consist of N atoms, initially at temperature T_1. The pressure and volume are then slowly doubled, in such a way as to trace out a straight line on the P-V diagram. In terms of N, k, and T_1, find

(a) the work done by the gas.

(b) If one defines an equivalent specific heat capacity ($c = \Delta Q/\Delta T$, where ΔQ is the total heat transferred to the gas) for this particular process for the above monatomic gas, what is its value?

Problem 6:

Consider the Lagrangian $L = m\dot{x}\dot{y} - m\omega_0^2 xy$.

(a) Write down Lagrange's equations associated with this Lagrangian and solve them. What physical system does this Lagrangian describe?

(b) Determine the Hamiltonian of the system.

(c) Define new generalized coordinates x' and y' such that

$x = 2^{-1/2}(x' + iy')$, $y = 2^{-1/2}(x' - iy')$.

Write down the Lagrangian and Lagrange's equations in terms of the new generalized coordinates and velocities and solve them.

(d) Express the total energy of the system in terms of the new generalized coordinates and velocities, assuming these coordinates are real.

Problem 7:

(a) Show that if two events are separated in space and time so that no signal leaving one event can reach the other, then there is an observer for whom the two events are simultaneous.

(b) Show that the converse is also true: if a signal can get from one event to the other, then no observer will find them simultaneous.

(c) Show if a signal can get from one event to the other, then there is an observer for whom the two events have the same space coordinates, i.e. for whom the two events "happen at the same place".

Problem 8:

Consider an electron in the field of a plane electromagnetic wave. Calculate the energy per unit time radiated by the electron. From this calculate the scattering cross section.