

# Spring 2011 Qualifying Exam

## Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

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### Physical Constants:

**Planck constant:**  $h = 6.62606896 \times 10^{-34}$  Js,  $\hbar = 1.054571628 \times 10^{-34}$  Js

**Boltzmann constant:**  $k_B = 1.3806504 \times 10^{-23}$  J/K

**Elementary charge:**  $e = 1.602176487 \times 10^{-19}$  C

**Avogadro number:**  $N_A = 6.02214179 \times 10^{23}$  particles/mol

**Speed of light:**  $c = 2.99792458 \times 10^8$  m/s

**Electron rest mass:**  $m_e = 9.10938215 \times 10^{-31}$  kg

**Proton rest mass:**  $m_p = 1.672621637 \times 10^{-27}$  kg

**Neutron rest mass:**  $m_n = 1.674927211 \times 10^{-27}$  kg

**Bohr radius:**  $a_0 = 5.2917720859 \times 10^{-11}$  m

**Compton wavelength of the electron:**  $\lambda_c = h/(m_e c) = 2.42631 \times 10^{-12}$  m

**Permeability of free space:**  $\mu_0 = 4\pi \times 10^{-7}$  N/A<sup>2</sup>

**Permittivity of free space:**  $\epsilon_0 = 1/\mu_0 c^2$

**Gravitational constant:**  $G = 6.67428 \times 10^{-11}$  m<sup>3</sup>/(kg s<sup>2</sup>)

**Stefan-Boltzmann constant:**  $\sigma = 5.670400 \times 10^{-8}$  W m<sup>-2</sup> K<sup>-4</sup>

**Wien displacement law constant:**  $\sigma_w = 2.8977685 \times 10^{-3}$  m K

**Units:** 1 cal = 4.186 J

1 foot = 0.3048 m

### Useful integral:

$$\int x \sin^2(x) dx = \frac{x^2}{4} - \frac{1}{4} x \sin(2x) - \frac{1}{8} \cos(2x)$$

## Section I:

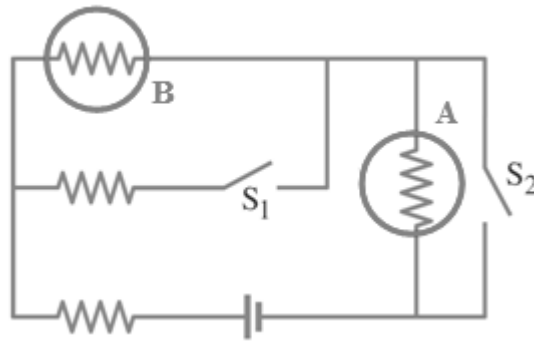
Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

### Problem 1:

An ideal radiator radiates with a total intensity of  $I = 5.68 \text{ kW/m}^2$ . At what wavelength does the spectral emittance  $I(\lambda)$  peak?  
(Give a numerical answer!)

### Problem 2:

Four identical light bulbs of resistance  $R$  are connected as shown in the figure.



The battery provides a potential difference  $V_0$ . The switches  $S_1$  and  $S_2$  can be open and/or closed in four different combinations: open-open, closed-closed, open-closed, closed-open.

- Consider light bulb A: determine which switch combinations would produce the brightest and dimmest light in bulb A.
- Consider light bulb B: determine which switch combinations would produce the brightest and dimmest light in bulb B.

**Problem 3:**

A space ship has a proper length of 100 m. It travels close to the Earth's surface with a constant speed of  $0.8c$ . Earth observers decide to measure the length of the ship by erecting two towers that coincide with the ends of the ship simultaneously (in the Earth's frame) as it passes by.

- (a) How far apart do the observers on Earth build the towers?
  - (b) How long do the observers on Earth say it takes for the nose of the ship to travel from tower A to tower B?
  - (c) How long, according to the measurements in the spaceship frame, does it take for the nose of the ship to travel from tower A to tower B?
  - (d) As measured by passengers in the spaceship, how far apart are the two towers?
  - (e) In the spaceship frame, how long does it take a beam of light to travel from the front to the rear of the spaceship?
  - (f) How much time, according to the observers on Earth, is required for a beam of light to travel from the front to the rear of the moving spaceship?
- (Give numerical answers!)

**Problem 4:**

A motorcyclist rides around the inside of a vertical cylinder of radius 50 feet. If the coefficient of friction is 0.5, calculate the minimum safe speed.

(Give a numerical answer!)

**Problem 5:**

Air at  $20.0^\circ\text{C}$  in the cylinder of a Diesel engine is compressed from an initial pressure of 1.00 atm and volume of  $800\text{ cm}^3$  to a volume of  $60\text{ cm}^3$ . Assume that air behaves as an ideal gas with  $\gamma = 1.40$  and that the compression is adiabatic. Find the final pressure and temperature of the air. (Give numerical answers!)

**Problem 6:**

An electromagnetic wave with frequency  $6.98 \times 10^{14}\text{ Hz}$  propagates with a speed of  $2.22 \times 10^8\text{ m/s}$  in a certain piece of glass.

- (a) Find the wavelength of the wave in the glass.
- (b) What is the wavelength of a wave with the same frequency propagating in vacuum?
- (c) What is the index of refraction  $n$  of the glass for an EM wave of this frequency?
- (d) What is the largest incident angle (with respect to the normal) for light of this frequency to be able to travel from glass into vacuum?

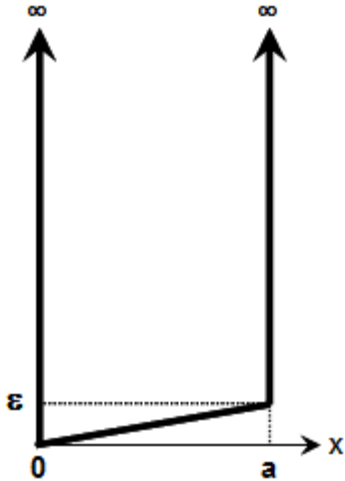
(Give numerical answers!)

**Problem 7:**

A particle of mass  $m$  moves in the potential

$$U(x) = \begin{cases} 0, & x < 0, x > a \\ \frac{\varepsilon x}{a}, & 0 \leq x \leq a \end{cases},$$

where  $\varepsilon$  is a small perturbation.

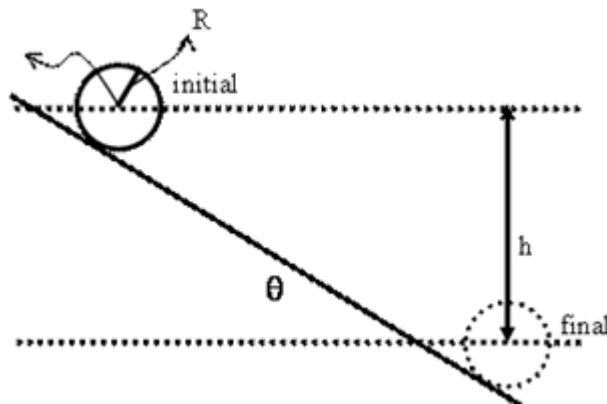


Use first-order perturbation theory to find the ground state energy of the particle.

**Problem 8:**

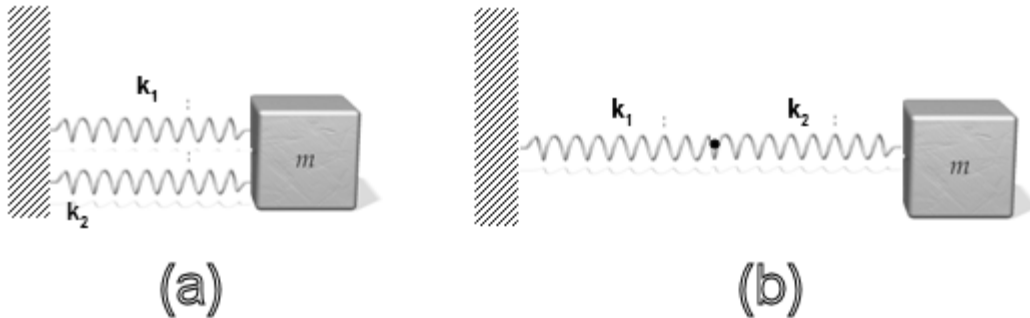
A circular hoop of mass  $M$  and radius  $R$  rolls down a plane inclined at an angle  $\theta$  in the Earth's gravitational field. Assume that the hoop starts from rest and rolls without slipping.

- What is the speed of the center of the hoop when it is a distance  $h$  below its initial position?
- What is its angular velocity when the center is a distance  $h$  below its initial position?
- How long does it take for the center to reach this position?



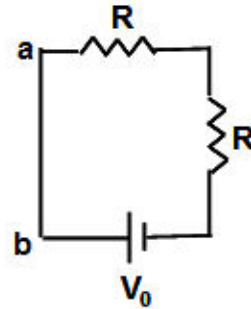
**Problem 9:**

Calculate the frequencies of oscillations of the mass  $m$  for the two spring configurations shown in the figures. The springs have elastic constants  $k_1$  and  $k_2$ .

**Problem 10:**

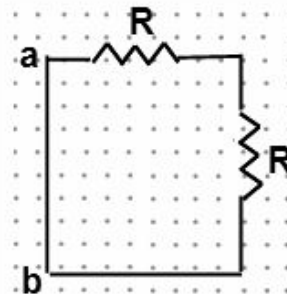
Consider the square circuit shown in the figure below, with a battery with potential  $V_0 = 1.5 \text{ V}$  and two identical resistors with  $R = 1 \text{ k}\Omega$ . Each side of the square has a length of  $50 \text{ cm}$ .

(a) Find the current  $I_0$  flowing in the circuit. Determine the potential of point a with respect to point b, i.e.  $V_{ab}$ . Does  $\int_c^a \vec{E} \cdot d\vec{r}$  depend on what path you take along the circuit?



(b) Now consider the same circuit but without the battery. A uniform magnetic field of magnitude  $B_0$  is perpendicular to the plane of the circuit and points out of the page. The circuit cannot be moved or deformed.

At what rate (magnitude and direction) does  $B$  have to change to produce the same current  $I_0$  as in the circuit in part (a)?



(c) When a current  $I_0$  is flowing in the circuit in part (b), does  $\int_c^a \vec{E} \cdot d\vec{r}$  depend on what path you take along the circuit? Explain!

## Section II:

Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

### Problem 11:

Consider the motion of a point of mass  $m$  subjected to a potential energy function of the form

$$U(x) = U_0 \left( 1 - \cos \left( \frac{x}{R_0} \right) \right) \text{ for } -\frac{\pi R_0}{2} < x < \frac{\pi R_0}{2}, \text{ where } x \text{ denotes distance, and } U_0 \text{ and } R_0 \text{ are}$$

positive constants with dimensions of energy and length, respectively.

- Find the position of stable equilibrium for the mass.
- Show that the motion of the mass in proximity of the stable equilibrium position is SHM.
- Find the period of the small oscillations.
- Find the period of the small oscillations for the same mass in the potential

$$U(x) = -U_0 \frac{1}{1 + \left( \frac{x}{R_0} \right)^2}.$$

### Problem 12:

Consider a quantum system with two possible orthonormal eigenstates of a physical property, for example, a single particle with isospin  $\frac{1}{2}$ . If the system is known with certainty to be in the first (or second) state, the state is represented by the eigenket  $|F\rangle$  (or  $|S\rangle$ ).

The most general state vector can be expressed as  $|\psi\rangle = |F\rangle\langle F|\psi\rangle + |S\rangle\langle S|\psi\rangle$ .

The quantum system can tunnel between the two states with a Hamiltonian

$$H = (|F\rangle\langle S| + |S\rangle\langle F|)\Delta E,$$

where  $\Delta E$  has the dimensions of energy.

- Find the normalized energy eigenkets and their corresponding energy eigenvalues.
- If the system is known with certainty to be in the first state  $|F\rangle$  at time  $t = 0$ , what is the probability for observing the system to be in the state  $|S\rangle$  at time  $t = t'$ ?

**Problem 13:**

Clever farmers whose lands are crossed by large power lines have been known to steal power by stringing wire near the power line and making use of the induced current.

Suppose that the farmer places a rectangular loop with two sides of length  $a$  parallel to the power line, the closest one at distance 5 m from it. The loop and the power line lie in the same plane.

Let the length of the sides perpendicular to the power line be  $b = 0.5$  m. The power line carries a 60 Hz alternating current with a peak current of 10 kA.

(a) If the farmer wants a peak voltage of 170 V (which is the peak of standard 120 V AC power) what should be the length  $a$ ?

(b) If the equipment the farmer connects to the loop has a resistance  $R = 5 \Omega$ , what is the farmer's average power consumption?

(c) If the power company charges 10 cents per kWh, what is the monetary value of the energy stolen each day?

(Give a numerical answer!)

**Problem 14:**

A particle of mass  $m$  starts at rest on top of a smooth fixed hemisphere of radius  $R$ . Find the normal force and determine the angle at which the particle leaves the hemisphere under the influence of gravity.

**Problem 15:**

The normalized wave function of an electron in a hydrogen atom, neglecting spin, is

$$\psi(\mathbf{r},t) = (1/3)^{1/2}\Phi_{100}(\mathbf{r},\theta,\phi) + (2/3)^{1/2}\Phi_{210}(\mathbf{r},\theta,\phi),$$

where the  $\Phi_{nlm}(\mathbf{r},\theta,\phi)$  are the usual normalized hydrogenic eigenfunctions.

- (a) If a single measurement is made of the energy, what results are possible?
- (b) What are the probabilities of obtaining each of the particular possible results?
- (c) What is the expectation value of the energy?
- (d) If a single measurement is made of the total angular momentum, what results are possible?
- (e) What are the probabilities of obtaining each of the particular possible results?
- (f) If the observable  $O$  is a constant of the motion, then it satisfies the equation

$$d/dt \langle O \rangle = \langle \partial O / \partial t \rangle + i/\hbar \langle [H, O] \rangle = 0$$

where  $H$  is the Hamiltonian.

For the hydrogen atom described above, is the  $z$ -component of angular momentum,  $L_z$ , a constant of the motion?

- (g) Is the  $z$ -component of linear momentum,  $p_z$ , a constant of the motion?
- (h) How do the expectation values of  $L_z$  and  $p_z$  depend on time?

Given:

$$\Phi_{100}(\mathbf{r},\theta,\phi) \propto \exp(-r/a_0), \quad \Phi_{210}(\mathbf{r},\theta,\phi) \propto \exp(-r/(2a_0)).$$

$$\int \Phi_{100}^* (\partial/\partial z) \Phi_{210} d^3r = (\sqrt{2}/a_0)(2/3)^4 \exp(-i(E_2 - E_1)t/\hbar) = A \exp(-i(E_2 - E_1)t/\hbar)$$

$$\int \Phi_{210}^* (\partial/\partial z) \Phi_{100} d^3r = -(\sqrt{2}/a_0)(2/3)^4 \exp(i(E_2 - E_1)t/\hbar) = -A \exp(i(E_2 - E_1)t/\hbar)$$