Fall 2011 Qualifying Exam

Part II

Mathematical tables are provided. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Mark exactly six problems.

Physical Constants:

Planck constant: $h = 6.62606896 \times 10^{-34}$ Js, $\hbar = 1.054571628 \times 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 \times 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 \times 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 \times 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 \times 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 \times 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 \times 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 \times 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 \times 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 \times 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 \times 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 \times 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 \times 10^{-3}$ m K

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

(a) A point charge q rests at the origin. A natural choice of potentials for this static problem is

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \quad \vec{A}(\vec{r},t) = 0$$

.

Consider the gauge transformation with

$$\lambda(\vec{r},t) = \frac{qt}{4\pi\varepsilon_0 r} + \frac{k}{r^3}$$

,

where *k* is a constant.

Calculate the transformed potentials and fields. Discuss the result.

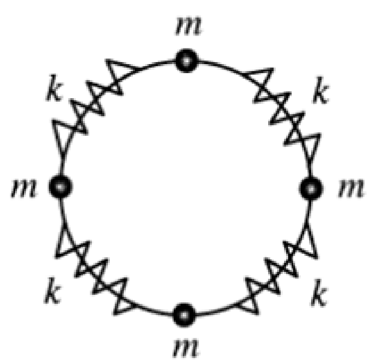
(b) Find **E** and **B** given the potentials $\vec{A}(\rho,\phi,z) = -\frac{I}{4\pi}\ln(\rho^2)\hat{z}$, V = 0. Which charge or current distributions produce these fields? Can you find a gauge transformation which leaves the transformed vector potential $\vec{A}'(\rho,\phi,z) = 0$?

Problem 2:

Four mass points of mass m move on a circle of radius R. Each mass point is coupled to its two neighboring points by a spring constant k.

(a) Find the Lagrangian of the system, and derive the equations of motion of the system.

(b) Calculate the eigen-frequencies of the system, and discuss the related eigen-vibrations.



Problem 3:
A proton and a neutron are confined by a three-dimensional potential. For this problem assume that the proton and neutron do not interact with each other, and neglect spin-orbit interactions. Both particles have spin ½. Including the spins, the ground state is four-fold degenerate. To this system we now add the interaction between the magnetic dipole moments of the particles described by the interaction Hamiltonian

$$H' = -k\boldsymbol{S}_p \cdot S_n,$$

where $\mathbf{S}_p$, $\mathbf{S}_n$ are the spin operators of the proton and neutron, respectively, and k is a positive constant.

(a) Consider the following operators:

$$S_p^2,\ S_n^2,\ S_{pz},\ S_{nz}, S^2, S_z.$$

where $\mathbf{S} = \mathbf{S}_p + \mathbf{S}_n$. State which of these operators commute with H'.

(b) Into how many distinct energy levels does the original ground state split in the presence of H'? Calculate the corresponding energies and state their degeneracy.

We now place the system into a uniform external magnetic field, which points in the positive z-direction, $\mathbf{B} = B\,\mathbf{k}$. The spin-spin interaction described by H' continues to be present and the additional interaction Hamiltonian is

$$H'_B = b\left(S_{pz} + S_{nz}\right)B_z$$

where b is a positive constant.

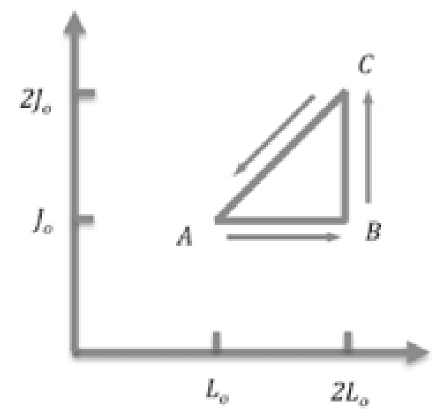
(c) Calculate the corrections to the energies of the states identified in part (b) due to the presence of the magnetic field.

(d) Sketch a graph of the energy levels as a function of the external magnetic field strength, B, including the effects of both H' and H'_B. Identify the curves with the corresponding states identified in part (b).

Problem 4:

A stretched rubber band contracts when heated under constant tension. Its temperature increases when stretched adiabatically. The equation of state for an idealized rubber band is $J = \alpha LT$, where J is the tension in the rubber band, L is its length, T is the absolute temperature and α is a constant. For reversible processes we have for the rubber band $TdS = dQ_{rev} = c_L\, dT - J\, dL$. The heat capacity of the band at constant length is c_L = constant.

Consider a heat engine that uses a rubber band in the three-step cycle shown.



Start with a stretched rubber band of length L_0, tension J_0, and temperature T_A.

Take the band through a sequence of reversible processes.

A → B:

The rubber band is stretched under constant tension J_0 to a length $2L_0$ while in contact with a heat reservoir of temperature T_B.

B → C:

While in contact with a heat reservoir of temperature T_C the tension of the rubber band is increased from J_0 to $2J_0$ at constant length $2L_0$.

C → A:

While in contact with a heat reservoir of temperature T_A, tension and length decrease linearly from $(2J_0, 2L_0)$ to (J_0, L_0).

(a) Find the ratios T_B/T_A and T_C/T_A. What is the ratio T_{hot}/T_{cold}?

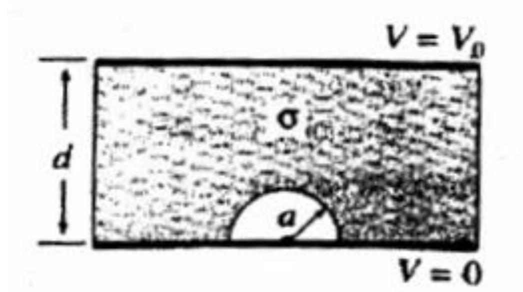
(b) Find the work done by the heat engine as it moves through one cycle A → B → C → A.

(c) During one cycle A → B → C → A, how much heat is extracted from the hot reservoir and how much heat is dumped into the cold reservoir?

(d) Find the efficiency of this rubber-band heat engine and compare it to the efficiency of a Carnot engine operating between the same temperatures.

Problem 5:

Two very large metal plates are held a distance d apart, one at potential zero and the other at potential V_0. A metal sphere of radius *a* is sliced in two, and one hemisphere is placed on the grounded plate, so that its potential is likewise zero. The radius of the sphere *a* is very small compare to the distance d between the plates, (*a* << d), so that you may assume that the electric field near the upper is constant. If the region between the plates is filled with a weakly conducting material of conductivity σ, what current flows to the hemisphere?



Problem 6:

At the beginning of the development of modern quantum mechanics, N. Bohr and A. Sommerfeld formulated a "quantization prescription" for periodic motion. Accordingly, only such trajectories in phase space are admitted for which the phase integral that extends over a period of motion $\oint p_\alpha dq_\alpha = n_\alpha h, \quad n_\alpha = 0, 1, 2 \ldots$ is a multiple of Planck's action quantum h. The generalized coordinates and canonically conjugate momenta are q_α , p_α , respectively.

(a) Write the Lagrangian, Hamiltonian, Hamilton equations, and constants of motion for a particle with mass m and potential energy $U(r) = -e^2/r$, with $e^2 = q_e^2/(4\pi\varepsilon_0)$.

(b) Apply the "quantization prescription" to p_r and p_φ, the conjugate momenta of the spherical coordinated r and φ.

(c) Calculate the bound energy states of the hydrogen atom from the quantization prescription.

Useful integral:

$$\int \frac{\sqrt{ar^2+br+c}}{r}\,dr = \sqrt{ar^2+br+c} - \frac{b}{2\sqrt{-a}}\arcsin\left(\frac{2ar+2b}{\sqrt{-\Delta}}\right) + \frac{c}{\sqrt{-c}}\arcsin\left(\frac{br+2c}{r\sqrt{-\Delta}}\right)$$

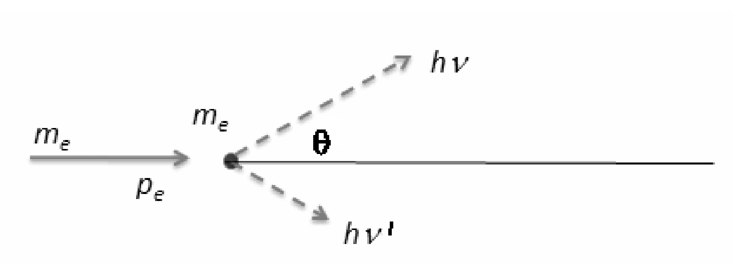
for $a < 0, c < 0, \Delta < 0$. Here $\Delta = 4ac - b^2$.

Problem 7:
The state of a free particle at time t is described by the following wave function:

$$\psi(x) = \begin{cases} 0 \text{ for } x < -b \\ A \text{ for } -b \le x \le 3b \\ 0 \text{ for } x > 3b \end{cases}$$

(a) Find A using the normalization condition. (You may choose the phase convention such that A is real.)
(b) What is the probability of finding the particle within the interval [0, b]?
(c) Calculate $\langle x \rangle$ and $\langle x^2 \rangle$ for this state.
(d) Calculate the momentum probability density.
(e) Calculate $\langle p \rangle$ for this state.

Problem 8:
An electron with mass m_e and momentum p_e hits a positron (same mass but opposite charge) at rest. They annihilate producing two photons. (Why couldn't they produce just one photon?)



(a) Following the collision, the scattered photon is deflected by an angle, θ. Find the photon's energy if it emerges at angle θ.
(b) What are the maximum and minimum energy an emitted photon can have, and at what emission angles do these occur?
(c) Express your result from part (b) in terms of the total energy E_{CM} available in the center of momentum frame and the speed v of the CM frame with respect to the laboratory frame. Interpret your result.