January 2016 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\lambda)) - 1]^{-1}$

Useful Integrals:

$\int dx\ \sqrt{x}/\sqrt{(b - x)} = \sqrt{(bx - x^2)} - ½b*\sin^{-1}((2x - b)/b)$

$\int x dx/(a - bx + cx^2)^{3/2} = (4a - 2bx)/[(b^2 - 4ac)(a - bx + cx^2)^{1/2}]$

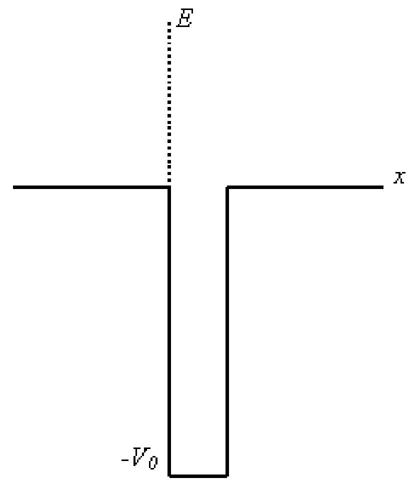
$$\int \sin(a x) \sin(b x)\, dx = \frac{\sin(x(a-b))}{2(a-b)} - \frac{\sin(x(a+b))}{2(a+b)} \qquad \int \cos(a x) \sin(b x)\, dx = \frac{\cos(x(a-b))}{2(a-b)} - \frac{\cos(x(a+b))}{2(a+b)}$$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

A particle of mass m is released a distance b from a fixed origin of force that attracts the particle according to the inverse square law $F(x) = -k/x^2$. Find the time required for the particle to reach the origin. Use this result to show that, if the Earth were suddenly stopped in its orbit, it would take approximately 65 days for it to collide with the Sun. Assume that the Sun is as a fixed point mass and Earth's orbit is circular.

Problem 2:



Consider a particle in the ground state of a one-dimensional square well of width a and depth V_0.
Assume that the well is very deep and

$$(2m(E + V_0)/(2mV_0))^{1/2} \ll 1$$

for the ground state, so that the ground state wave function is nearly identical to that of the infinite square well.
At t = 0, a time dependent perturbation $W(t) = W\cos\omega t$ is turned on.
What is the minimum frequency necessary to free the particle from the well? For frequencies greater than this minimum frequency, use perturbation theory to find the transition rate.
You can assume that the free particles will be in a box of size $L \gg a$.

Problem 3:

A model of the hydrogen atom was proposed before the advent of quantum mechanics, which consists of a single electron of mass m and an immobile uniform spherical distribution of positive charge with radius R. Assume that the positive charge interacts with the electron via the usual Coulomb interaction but otherwise does not offer any resistance to the motion of the electron.

(a) Explain why the electron's equilibrium position is at the center of the positive charge.

(b) Show that the electron will undergo simple harmonic motion if it is displaced a distance $d < R$ away from the center of the positive charge. Calculate its frequency of oscillation.

(c) How big would the atom need to be in order to emit red light with a frequency of $4.57 \cdot 10^{14}$ Hz? Compare your answer with the radius of the hydrogen atom.

(d) If the electron is displaced a distance $d > R$ from the center, will it oscillate in position? Will it undergo simple harmonic motion? Explain!

Problem 4:

Assume magnetic charges exist and Maxwell's equations are of the form

$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$, $\nabla\cdot\mathbf{B} = \mu_0\rho_m$, $-\nabla\times\mathbf{E} = \mu_0\mathbf{j_m} + \partial\mathbf{B}/\partial t$, $\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\partial\mathbf{E}/\partial t$.

Assume a magnetic monopole of magnetic charge q_m is located at the origin, and an electric charge q_e is placed on the z-axis at R distance a from it.

(a) Write down expressions for the electric field **E**(**r**) and the magnetic field **B**(**r**). Make a sketch.

(b) Write down expressions for the momentum density **g**(r) and angular momentum density $\boldsymbol{\mathcal{L}}(\mathbf{r})$ of the electromagnetic field.

(c) Show that there is electromagnetic angular momentum L_z about the z-axis and derive an expression for it.

Useful vector identity: $(\mathbf{a}\cdot\nabla)\mathbf{n} = (1/r)[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})]$. Here **n** is **r**/r is the unit radial vector.

Problem 5:

Consider a particle of mass m in an one-dimension infinite square well of length L. Assume that the particle is in the nth eigenstate (n = 1, 2, 3, … ,).

(a) The momentum is measured. Show that the probability distribution $P_n(k)$ for measuring a momentum $p = \hbar k$ is $P_n(k) = [2\pi L n^2 (k^2L^2 - n^2\pi^2)^2]\,[1 + (-1)^{n+1}\cos(kL)]$.

(b) What outcome of a momentum measurement is most likely? Does your result agree with your intuition?

(c) Which momenta cannot be the result of a momentum measurement? Why is that so?

Problem 6:

In the following, two examples of a decaying system are presented, where the decay products travel with velocities comparable to the speed of light c.

1. Two electrons are ejected simultaneously in opposite directions from an atom. Each electron has a speed as measured by a laboratory observer of 0.5 c. What is the speed of one electron as seen from the rest frame of the other electron

(a) in the non-relativistic approach?

(b) in the relativistic approach?

Distinguish carefully the velocities in the respective frames.

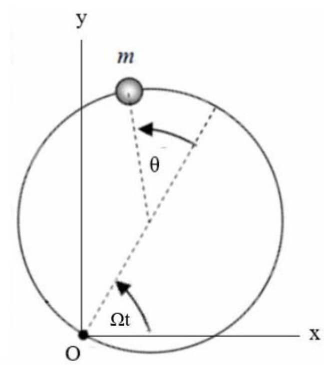
2. The neutral pi meson, π°, has a rest mass of 135 MeV/c^2. It decays into two photons (γ rays) of equal energy and opposite direction in its rest frame. In the laboratory frame the π° is moving with a total energy 25% larger than its rest energy.

(a) What are the energies of the γ rays, as measured in the laboratory, if the decay process causes them to be emitted in opposite directions along the pion's original line of motion?

(b) What is the velocity of each γ ray as observed by the other?

Problem 7:

Consider a bead of mass m sliding freely on a smooth circular wire of radius b which rotates in a horizontal plane about one of its points O, with constant angular velocity Ω. Let θ be the counterclockwise angle between the diameter that passes through the mass and the diameter that passes through the point O, with $\theta = 0$ the case where the mass is farthest from O.



(a) Find the equation of motion for θ. Compare this equation with the equation of motion for a simple pendulum (point mass and massless rod).

(b) For the initial conditions $\theta = 0$, $d\theta/dt = \omega_0$ at $t = 0$, describe the θ motion that occurs for $|\omega_0| < 2\Omega$ and for $|\omega_0| > 2\Omega$. (Note: The same equations have the same solutions.)

(c) Describe the θ motion that occurs for $|\omega_0| << 2\Omega$.

Problem 8:

A muonic atom is one in which an atomic electron is replaced by a muon. The muon is 209 times more massive than the electron.

(a) Compute the energy of the 2p – 1s muonic transition in ^{208}Pb (Z = 82) under the assumption that Pb is a point nucleus. Make reasonable assumptions and explain your assumptions. Compare your result with the observed value of 5.8 MeV.

(b) Use the transition-energy values computed and given in part (a) and simple scaling rules for hydrogenic atoms to give an order-of-magnitude estimate of the nuclear radius of Pb (whose actual nuclear charge radius is ~ 6 fm).

(c) Use perturbation theory to calculate the first-order shift in the ground-state energy of an **electron** in hydrogenic ^{208}Pb (Z = 82) due to the finite size of this nucleus. Assume the nucleus is a uniformly charged sphere. Why is this not a valid approach for the muonic Pb atom?

The ground state wave function of the hydrogen atom is $|1,0,0\rangle = (\pi a_0^3)^{-1/2} \exp(-r/a_0)$.