January 2014 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Consider a two-dimensional infinite potential square well of width L,
(V = 0 for 0 < x, y < L, V = infinite everywhere else) with an added perturbation
$H' = g\, sin\left(\frac{2\pi x}{L}\right) sin\left(\frac{2\pi y}{L}\right)$.

(a) Calculate the first order perturbation to the ground state energy eigenvalue.
(b) Calculate the first order perturbation to the first excited state energy eigenvalue

Solution:
Stationary perturbation theory for non-degenerate and for degenerate states
The ground state of two-dimensional infinite potential square well is not degenerate. The first excited state is two-fold degenerate.
$H = H_0 + H'$.
The eigenfunctions of H_0 are $\Phi_{nl}(x,y) = (2/L)\sin(n\pi x/L)\sin(l\pi y/L)$, with eigenvalues
$E = (n^2 + l^2)\pi^2\hbar^2/(2mL^2)$.
(a) For the ground state n = l = 1. The ground state is not degenerate.
The unperturbed ground state energy is $E_0^0 = \hbar^2/(mL^2)$. The first order perturbation correction is
$E_1^0 = \langle\Phi_{11}|H'|\Phi_{11}\rangle = (g4/L^2)\int_0^L \sin^2(\pi x/L)\sin^2(\pi y/ L)\sin(2\pi x/L)\sin(2\pi y/L)dxdy$.
$= (4g/L^2)\int_0^L \sin^2(\pi x/L)\sin(2\pi x/L)dx)\int_0^L \sin^2(\pi y/L)\sin(2\pi y/L)dy = 0$.

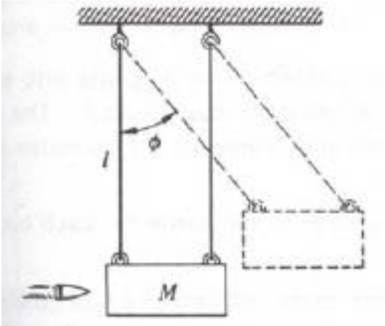
(b) The first excited state is two-fold degenerate.
We can have n = 2, l = 1, or n = 1, l = 2.
We have to diagonalize the matrix of H' in the subspace spanned by these two degenerate states.

$$\begin{pmatrix} <\Phi_{12}|H'|\Phi_{12}> -E & <\Phi_{12}|H'|\Phi_{21}> \\ <\Phi_{21}|H'|\Phi_{12}> & <\Phi_{21}|H'|\Phi_{21}> -E \end{pmatrix} = 0$$

$\langle\Phi_{12}|H'|\Phi_{12}\rangle = (4g/L^2)\int_0^L \sin^2(\pi x/L)\sin(2\pi x/L)dx)\int_0^L \sin^2(2\pi y/L)\sin(2\pi y/L)dy = 0$.
$\langle\Phi_{21}|H'|\Phi_{21}\rangle = (4g/L^2)\int_0^L \sin^2(2\pi x/L)\sin(2\pi x/L)dx)\int_0^L \sin^2(\pi y/L)\sin(2\pi y/L)dy = 0$.
$\langle\Phi_{12}|H'|\Phi_{21}\rangle = (4g/L^2)\int_0^L \sin(\pi x/L)\sin^2(2\pi x/L)dx)\int_0^L \sin^2(2\pi y/L)\sin(\pi y/L)dy = C$
$C = (4g/\pi^2)(\int_0^\pi \sin(x)\sin^2(2x)dx)^2 = (4g/\pi^2)*64/225$.
$\langle\Phi_{12}|H'|\Phi_{21}\rangle = (4g/L^2)\int_0^L \sin(\pi x/L)\sin^2(2\pi x/L)dx)\int_0^L \sin^2(2\pi y/L)\sin(\pi y/L)dy = C$
$E^2 - C^2 = 0$. $E_1^1 = \pm C$. The perturbation removes the degeneracy.

Problem 2:

A simple way to measure the speed of a bullet is with a ballistic pendulum. As illustrated, this consists of a wooden block of mass M into which the bullet is shot horizontally. The block is suspended from cables of length l, and the impact of the bullet causes it to swing through a maximum angle ϕ, as shown. The initial speed of the bullet is v and its mass is m.



(a) What is the speed of the block V immediately after the bullet comes to rest inside the wooden block? (Assume that this happens quickly)

(b) Find an expression for the speed v of the bullet in terms of the quantities that can be easily measured m, M, g, l, and ϕ.

Solution:

a) Use momentum conservation for the first part of this problem. Since the wodden block is initially at rest the initial momentum of the system is

$$P_{initial} = mv$$

The collision between the bullet and the wodden block is completely inelastic, so after the collision they will have the same velocity V, so the final momentum is

$$P_{final} = (m+M)V$$

Momentum conservation implies

$$mv = (m+M)V \quad \Rightarrow \quad \boxed{V = \frac{m}{m+M}v}$$

b) We can now use mechanical energy conservation to solve the second part of the problem. At the moment the bullet has come to rest inside the wodden block the total mechanical energy of the system is (we assume the potential energy is zero at the initial position of the wodden block)

$$E_1 = T + V = \frac{1}{2}(m+M)V^2 + 0 = \frac{1}{2}(m+M)\left(\frac{m}{m+M}v\right)^2 = \frac{1}{2}\frac{m^2}{M+m}v^2$$

When the wodden block + bullet reaches it maximum height (as shown on the figure) is has no kinetic energy, so its total mechanical energy E_2 is given by the gravitational potential energy

$$E_2 = (m+M)gh = (m+M)g(l - l\cos\phi) = (m+M)gl(1-\cos\phi)$$

Mechanical energy conservation implies

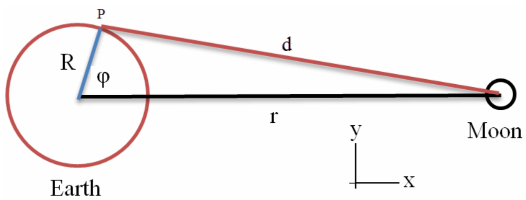
$$E_1 = E_2 \quad \Rightarrow \quad \frac{1}{2}\frac{m^2}{M+m}v^2 = (m+M)gl(1-\cos\phi) \quad \Rightarrow \quad v^2 = 2\frac{(m+M)^2}{m^2}gl(1-\cos\phi) \quad \Rightarrow$$

$$\boxed{v = \frac{m+M}{m}\sqrt{2gl(1-\cos\phi)}}$$

Problem 3:

The tidal force $\Delta\mathbf{F}_\varphi/m$ of the Moon on the Earth is defined as the difference between the gravitational field of the Moon at a point on the Earth's surface defined by the angle φ and the gravitational field of the Moon at the Earth's center. (See figure.) Derive an expression for $\Delta\mathbf{F}_\varphi/m$. Simplify by assuming that R, the radius of the Earth, is much smaller than r, the Earth-Moon distance.

Find the x- and y components and the tangential and radial components of $\Delta\mathbf{F}_\varphi/m$.



Solution:

Newton's law of gravity

At the point P $\Delta\mathbf{F}_\varphi/m = (GM/d^2)(\mathbf{e}_x(r - R\cos\varphi)/d - \mathbf{e}_y(R\sin\varphi)/d) - (GM/r^2)\mathbf{e}_x$

Here M is the mass of the moon.

$d^2 = (r - R\cos\varphi)^2 + (R\sin\varphi)^2 = r^2 + R^2 - 2rR\cos\varphi$

$\Delta\mathbf{F}_\varphi/m = (GM/(r^2 + R^2 - 2rR\cos\varphi)^{3/2})[\mathbf{e}_x(1 - (R/r)\cos\varphi) - \mathbf{e}_y(R/r)\sin\varphi)] - (GM/r^2)\mathbf{e}_x$

$= -(GM/r^2))\mathbf{e}_x[(1 - (R/r)\cos\varphi)/(1 + R^2/r^2 - 2(R/r)\cos\varphi)^{3/2} - 1]$

$-(GM/r^2))\mathbf{e}_y(R/r)\sin\varphi)/(1 + R^2/r^2 - 2(R/r)\cos\varphi)^{3/2}$

Since r << R, we expand and neglect terms higher than first order in R/r.

$\Delta\mathbf{F}_\varphi/m = (GM/r^2))\mathbf{e}_x[(1 - (R/r)\cos\varphi)/(1 - 3(R/r)\cos\varphi) - 1]$

$-(GM/r^2))\mathbf{e}_y(R/r)\sin\varphi)$

$(\Delta\mathbf{F}_\varphi/m)_x = (GM/r^2)[(1 - (R/r)\cos\varphi)/(1 - 3(R/r)\cos\varphi) - 1] = (2GMR/r^3)\cos\varphi$

$(\Delta\mathbf{F}_\varphi/m)_y = -(GMR/r^3)\sin\varphi$

We have azimuthal symmetry about the x-axis.

$(\Delta\mathbf{F}_\varphi/m)_{tangential} = (\Delta\mathbf{F}_\varphi/m)_x\sin\varphi - (\Delta\mathbf{F}_\varphi/m)_y\cos\varphi = (GMR/r^3)3\cos\varphi\sin\varphi = (GMR/r^3)(3/2)\sin(2\varphi)$

A positive $(\Delta\mathbf{F}_\varphi/m)_{tangential}$ points clockwise.

$(\Delta\mathbf{F}_\varphi/m)_{radial} = (\Delta\mathbf{F}_\varphi/m)_x\cos\varphi + (\Delta\mathbf{F}_\varphi/m)_y\sin\varphi = (GMR/r^3)(2\cos^2\varphi - \sin^2\varphi)$

$= (GMR/r^3)(2 - 3\sin^2\varphi) = ½(GMR/r^3)(1 + 3\sin(2\varphi))$

Problem 4:

Most of neutrinos from the Sun are produced in the chain of processes called the “pp-cycle”:

(1) $p + p \rightarrow {}^2H + \beta^+ + \nu$

(2) $p + {}^2H \rightarrow {}^3He + \gamma$

(3) ${}^3He + {}^3He \rightarrow {}^4He + p + p$

Estimate the order of magnitude of the neutrino flux (neutrinos/(cm^2 s) from these reactions on Earth using the following data:

Earth-Sun distance = $1.5*10^{11}$ m

proton mass = 938.272 MeV/c^2

^{4}He mass = 3727.379 MeV/c^2

Assume the Earth can be modelled as a black body with temperature T = 300K. On average, it emits as much radiation as it receives from the Sun. Use this to estimate the energy flux from the sun on Earth.

Solution:

Nuclear energy, radiation laws

Producing one ^{4}He nucleus releases 2 neutrinos. How much energy is released in the production of one ^{4}He?

$E = 4m(p)c^2 - m({}^4He)c^2 = 25.7$ MeV (The positrons will annihilate with electrons, thus producing gammas (energy).

Thus 2 neutrinos are produced for every 25.7 MeV of energy that is released.

What is the solar energy flux at the sun-earth distance?

We assume that it is equal to the power per unit area emitted by Earth.

$I = \sigma T^4 = (5.67 * 10^{-8}\ W\ m^{-2}\ K^{-4})*(300\ K)^4 = 459\ W/m^2 = 2.87*10^{15}\ MeV/(m^2\ s)$

We therefore expect a neutrino flux of approximately $2* 2.87*10^{15} /25.7$ neutrinos/(m^2 s),

or $2.23*10^{14}$ neutrinos/(m^2 s) = $2.23*10^{10}$ neutrinos/(cm^2 s).

(The energy flux is underestimated, the real neutrino flux is ~$6*10^{10}$ neutrinos/(cm^2 s).)

Problem 5:

(a) From Maxwell's equations, derive the conservation of energy equation,

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J} \cdot \boldsymbol{E} \ ,$$

where u is the energy density and **S** is the Poynting vector.

(b) A long, cylindrical conductor of radius a and conductivity σ carries a constant current I. Find **S** at the surface of the cylinder and interpret your result in terms of conservation of energy.

Solution:

Maxwell's equations, energy density and energy flux in the electromagnetic field

(a) Maxwell's equations in SI units are

$$\vec{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \vec{\nabla} \cdot \boldsymbol{B} = 0, \quad \vec{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}.$$

$\mathbf{E}\cdot\mathbf{j} = \mathbf{E}\cdot(1/\mu_0)(\nabla\times\mathbf{B}) - \varepsilon_0(\partial\mathbf{E}/\partial t)\cdot\mathbf{E}$

$= -(1/\mu_0)\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \mathbf{B}\cdot(1/\mu_0)(\nabla\times\mathbf{E}) - \varepsilon_0(\partial\mathbf{E}/\partial t)\cdot\mathbf{E}$

$= -(1/\mu_0)\nabla\cdot(\mathbf{E}\times\mathbf{B}) - (1/\mu_0)\mathbf{B}\cdot(1/\mu_0)(\partial\mathbf{B}/\partial t)$

$= -(1/\mu_0)\nabla\cdot(\mathbf{E}\times\mathbf{B}) - (\partial/\partial t)((1/2\mu_0)B^2 + (\varepsilon_0/2)E^2)$.

$(\partial/\partial t)((1/2\mu_0)B^2 + (\varepsilon_0/2)E^2) + (1/\mu_0)\nabla\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot\mathbf{j}.$

Interpretation:

Let $u = (1/2\mu_0)B^2 + (\varepsilon_0/2)E^2$ be the energy density in the electromagnetic field.

Let $\mathbf{S} = (1/\mu_0)(\mathbf{E}\times\mathbf{B})$ be the energy flux.

Then $(\partial u/\partial t) + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{j}$.

$$\int_V \nabla\cdot\mathbf{S}\,dV = \oint_A \mathbf{S}\cdot\hat{\mathbf{n}}\,dA, \quad -\int_V \frac{\partial u}{\partial t}\,dV = \oint_A \mathbf{S}\cdot\hat{\mathbf{n}}\,dA + \int_V \mathbf{E}\cdot\mathbf{j}\,dV.$$

This is a statement of energy conservation. The rate at which the field energy in a volume decreases equals the rate at which it leaves across the boundaries plus the rate at which it gets converted into other forms.

(b) In the wire $\mathbf{j} = \sigma\mathbf{E}$. Let $\mathbf{j} = j\,\mathbf{e}_z$, $I = j\pi a^2 = \sigma E\pi a^2$. $E = I/(\sigma\pi a^2)$.

On the surface of the wire $\mathbf{E} = I/(\sigma\pi a^2)\,\mathbf{k}$.

Outside of the wire $\mathbf{B} = \mathbf{e}_\varphi\,\mu_0 I/(2\pi r)$. On the surface $\mathbf{B} = \mathbf{e}_\varphi\,\mu_0 I/(2\pi a)$.

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B}).$$

$\mathbf{S} = -\mathbf{e}_\rho\, I^2/(2\sigma\pi^2 a^3)$.

Energy is flowing into the wire and is converted into heat.

[The field energy that is flowing into the wire per unit length is $P = S2\pi a = I^2/(\sigma\pi a^2)$.

The thermal energy dissipated per unit length is $P = I^2R_{\text{unit length}}$.

$R_{\text{unit length}} = 1/(\sigma\pi a^2)$.

Therefore thermal energy dissipated per unit length = field energy flowing into the wire per unit length.]

Problem 6:

A wave packet for a quantum mechanical particle of mass m in one dimension is described by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{+R} dk\, \phi(k) \exp(i[kx - \omega(k)t])\ ,$$

where $\phi(k) \propto exp\left(-\frac{(k-k_0)^2}{4(\Delta k)^2}\right)$ is a "strongly peaked" distribution around $k = k_0$ with $\Delta k \Delta x = ½$ at $t = 0$.

(a) Show that $\psi(x,0) = \frac{1}{\sqrt{\Delta x\sqrt{2\pi}}} exp(ik_0 x) exp\left(\frac{-x^2}{4(\Delta x)^2}\right)$ (evaluate the integral),

and find the probability density $|\psi(x, 0)|^2$ of the particle. Use $\Delta k \Delta x = ½$ to eliminate Δk. Show your work!

(b) For a free particle $E = \hbar\omega = \hbar^2 k^2/(2m)$. Show that for a free particle

$$|\psi(x,t)|^2 = \frac{1}{\Delta x(t)\sqrt{2\pi}} exp\left(\frac{-\left(x-\frac{\hbar k_0 t}{m}\right)^2}{2(\Delta x(t))^2}\right), \text{ with } \Delta x(t) = \sqrt{(\Delta x)^2 + \frac{\hbar^2 t^2}{4m^2(\Delta x)^2}}\ .$$

(c) Determine the group velocity of the wave packet.

(d) Evaluate the time it takes for the wave-packet to double in spatial extent, specifically if the particle is an electron and $\Delta x \sim 10$ nm, at $t = 0$.

Useful integrals:

$\int_{-\infty}^{+\infty} \exp(-\alpha x^2)dx = \sqrt{(\pi/\alpha)}$

$\int_{-\infty}^{+\infty} \exp(-\alpha x^2 + i\beta x))dx = (\pi/\alpha)^{1/2}\exp(-\beta^2/(4\alpha))$

Note: You can complete some parts of the problem by using the given results without evaluating the integrals.

Solution:

(a) Normalize $\phi(k)$:

$N^2\int_{-\infty}^{+\infty}\exp(-(k - k_0)^2/(2(\Delta k)^2))dk = 1.$ $\int_{-\infty}^{+\infty}\exp(-(k - k_0)^2/(2(\Delta k)^2))dk = \Delta k\sqrt{(2\pi)}.$

$N = 1/(\Delta k\sqrt{(2\pi)})^{1/2}.$

$\psi(x, 0) = [\exp(ik_0x)N/\sqrt{(2\pi)}\ ]\int_{-\infty}^{+\infty}\exp(-(k - k_0)^2/(4(\Delta k)^2) + ix(k - k_0))dk$

$= [(\Delta k)^{1/2}2^{1/4}/\pi^{1/4}]\exp(ik_0x)\exp(-x^2(\Delta k)^2) = [\Delta x^{-1/2}(2\pi)^{-1/4}]\ \exp(ik_0x)\exp(-x^2/(4\Delta x^2))$

$|\psi(x, 0)|^2 = [1/(\Delta x(2\pi)^{1/2})]\ \exp(-x^2/(2\Delta x^2))$

This is a Gaussian with $\sigma = \Delta x$ centered at $x = 0$.

(b) $\psi(x, t) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \phi(k, t) \exp(ikx)\, dk$
$\phi(k, t) = N \exp(-(k - k_0)^2/(4(\Delta k)^2) - i\hbar k^2 t/(2m))$
$N = 1/(\Delta k\sqrt{(2\pi)})^{1/2}$.
$-(k - k_0)^2/(4(\Delta k)^2) + ixk - i\hbar k^2 t/(2m)$
$= -(k - k_0)^2/(4(\Delta k)^2) + ix(k - k_0) - i\hbar(k - k_0)^2 t/(2m) + i\hbar k_0^2 t/(2m) - i\hbar k k_0 t/m + ik_0 x$
$= -(k - k_0)^2[1/(4(\Delta k)^2) + i\hbar t/(2m)] + i(k - k_0)[x - \hbar k_0 t/m] + i\hbar k_0^2 t/(2m) - i\hbar k k_0^2 t/m + ik_0 x$
$= -(k - k_0)^2[\Delta x^2 + i\hbar t/(2m)] + i(k - k_0)[x - \hbar k_0 t/m] - i\hbar k_0^2 t/(2m) + ik_0 x$
$\psi(x, t) = \Delta x^{1/2} [(\Delta x^2 + i\hbar t/(2m))]^{-1/2}(2\pi)^{-1/4}]*$
$\exp(ik_0 x - i\hbar k_0^2 t/(2m)) \exp[-¼(x - \hbar k_0 t/m)^2/(\Delta x^2 + i\hbar t/(2m))]$
$|\psi(x, t)|^2 = \exp[-½ (x - \hbar k_0 t/m)^2/\Delta x(t)^2]/[(2\pi)^{1/2} \Delta x(t)]$, with $\Delta x(t) = (\Delta x^2 + \hbar^2 t^2/(4m^2\Delta x^2))^{1/2}$.

(c) $\psi(x, t)|^2$ is a Gaussian with $\sigma = \Delta x(t)$ centered at $x = \hbar k_0 t/m$. The center of the wave packet moves with speed $\hbar k_0 t/m$. This is the group velocity $vg = d\omega/dk$ of the wave packet.

(d) $\Delta x(t) = (\Delta x^2 + \hbar^2 t^2/(4m^2\Delta x^2))^{1/2} = 2\Delta x$
$\Delta x^2 + \hbar^2 t^2/(4m^2\Delta x^2) = 4\Delta x^2$. $t^2 = 12\ \Delta x^4\ m^2/ \hbar^2$.
$t^2 = 12*10^{-32}*(9.1*10^{-31})^2/(1.05*10^{-34})^2\ s^2 = \sim 10^{-23}\ s^2$, $t \sim 3*10^{-12}$ s.
We get the same order of magnitude result by just using the uncertainty principle.

Problem 7:

Calculate the force, as observed in the laboratory, between two electrons moving side by side along parallel paths 1 mm apart, if they have a kinetic energy of 1 eV and 1 MeV.

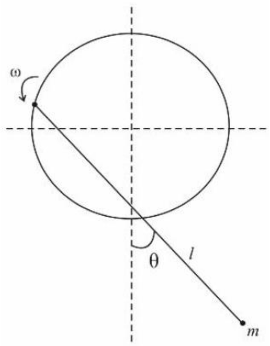
Solution:
We can calculate the force between the electrons in the rest frame of the electrons from $\mathbf{F} = q\mathbf{E}$ and transform this force to the laboratory frame, or we can calculate the electric field of one of the electrons in its rest frame, transform it into and electric and magnetic field in the laboratory frame, and calculate the force between the electrons from $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$.

Assume the electrons move with velocity $v\mathbf{i}$ in the x-direction in the laboratory frame K. Electron 1 has coordinates $y = z = 0$ and electron 2 has coordinates $y = y_0 = 1$ mm, $z = 0$.
In the rest frame K' of the electrons the force electron 1 exerts on electron 2 is
$\mathbf{F}' = \mathbf{j}\ kq_e^2/y'^2_0 = \mathbf{j}\ kq_e^2/y_0^2$.
$\mathbf{F}' = \mathbf{j}\ 9*10^9*(1.6*10^{-19})^2/10^{-6}$ N $= \mathbf{j}\ 2.3*10^{-22}$ N
$\mathbf{F}' = d\mathbf{p}'/dt' = d\mathbf{p}'/d\tau$, where τ is the proper time, a Lorentz invariant.
In K the force on the charge is $\mathbf{F} = d\mathbf{p}/dt = (1/\gamma)d\mathbf{p}/d\tau$, $\gamma\mathbf{F} = d\mathbf{p}/d\tau$.
Here $\gamma = (1 - \beta^2)^{-1/2}$, $\boldsymbol{\beta} = v\mathbf{i}/c$.
From the transformation properties of the momentum 4-vector
$p^\mu = (\gamma mc,\gamma m\mathbf{v}) = (E/c,\mathbf{p})$ = under a Lorentz transformation,
$\mathbf{p}_{\|}= \gamma(\mathbf{p}'_{\|} + \beta E'/c)$, $\mathbf{p}_\perp = \mathbf{p}'_\perp$, we have
$\gamma\mathbf{F}_\perp = d\mathbf{p}_\perp/d\tau = d\mathbf{p}'_\perp/d\tau = d\mathbf{p}'_\perp/dt' = \mathbf{j}\ kqe^2/y_0^2$, $\mathbf{F}_\perp = \mathbf{j}\ (1/\gamma)kq_e^2/y_0^2$.
$\gamma\mathbf{F}_{\|} = d\mathbf{p}_{\|}/d\tau = \gamma d\mathbf{p}'_{\|}/d\tau = \gamma d\mathbf{p}'_{\|}/dt' = 0$.
We have used that in K' $d(E'/c)/dt' = (1/c)dE'/dt' = 0$ since K' is the rest frame of the charges.
The force between the electrons in the lab frame is $\mathbf{F}_\perp = \mathbf{j}\ (1/\gamma)kq_e^2/y_0^2$.
1 eV $= (1/2)mv^2$, $\gamma = 1$, $\mathbf{F} = \mathbf{j}\ 2.3*10^{-22}$ N.
1 MeV $= (\gamma - 1)mc^2$, $(\gamma - 1) = 1.96$, $\gamma = 2.96$, $\mathbf{F} = \mathbf{j}\ 7.8*10^{-23}$ N.

Problem 8:

A simple pendulum of length l (massless string) and mass m is suspended from a pivot point on the circumference of a thin massless disc of radius a that rotates with a constant angular velocity ω about its central axis as shown in the figure.

Find the equation of motion of the mass m in terms of the generalized variable θ.



Solution:

Lagrangian Mechanics

The coordinates of the pivot point will be

$$(a\cos\omega t, a\sin\omega t)$$

so the coordinates (x, y) of the mass m will be

$$\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} + l\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

and its velocity will be

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = a\omega\begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix} + l\dot{\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

The kinetic energy T of the pendulum will be

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \tfrac{1}{2}m((-a\omega\sin\omega t + l\dot{\theta}\cos\theta)^2 + (a\omega\cos\omega t + l\dot{\theta}\sin\theta)^2)$$

$$=$$

$$\tfrac{1}{2}m(a^2\omega^2\sin^2\omega t + l^2\dot{\theta}^2\cos^2\theta - 2a\omega l\dot{\theta}\sin\omega t\cos\theta + a^2\omega^2\cos^2\omega t + l^2\dot{\theta}^2\sin^2\theta + 2a\omega l\dot{\theta}\cos\omega t\sin\theta$$

$$= \tfrac{1}{2}m(a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}(-\sin\omega t\cos\theta + \cos\omega t\sin\theta))$$

$$= \tfrac{1}{2}m(a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}\sin(\theta - \omega t))$$

and the potential energy V will be

$$V \quad = mgy = mg(a\sin\omega t - l\cos\theta)$$

So the Langrangian L is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}m(a^2\omega^2 + l^2\dot\theta^2 + 2a\omega l\dot\theta\sin(\theta-\omega t)) - mg(a\sin\omega t - l\cos\theta) \end{aligned}$$

In order to find Lagrange's equation we need

$$\begin{aligned} \tfrac{\partial L}{\partial\theta} &= \tfrac{\partial}{\partial\theta}(\tfrac{1}{2}m(a^2\omega^2 + l^2\dot\theta^2 + 2a\omega l\dot\theta\sin(\theta-\omega t)) - mg(a\sin\omega t - l\cos\theta)) \\ &= ma\omega l\dot\theta\cos(\theta-\omega t) - mgl\sin\theta \\ \tfrac{\partial L}{\partial\dot\theta} &= \tfrac{\partial}{\partial\dot\theta}(\tfrac{1}{2}m(a^2\omega^2 + l^2\dot\theta^2 + 2a\omega l\dot\theta\sin(\theta-\omega t)) - mg(a\sin\omega t - l\cos\theta)) \\ &= ml^2\dot\theta + ma\omega l\sin(\theta-\omega t) \\ \tfrac{d}{dt}\tfrac{\partial L}{\partial\dot\theta} &= ml^2\ddot\theta + ma\omega l\cos(\theta-\omega t)(\dot\theta-\omega) \end{aligned}$$

so the equation of motion (= Lagrange's equation) will be

$$\tfrac{\partial L}{\partial\theta} - \tfrac{d}{dt}\tfrac{\partial L}{\partial\dot\theta} = 0 \quad \Rightarrow$$

$$ma\omega l\dot\theta\cos(\theta-\omega t) - mgl\sin\theta - ml^2\ddot\theta - ma\omega l\cos(\theta-\omega t)(\dot\theta-\omega) = 0 \quad \Rightarrow$$

$$a\omega\dot\theta\cos(\theta-\omega t) - gl\sin\theta - l^2\ddot\theta - a\omega\dot\theta\cos(\theta-\omega t) + a\omega^2 l\cos(\theta-\omega t) = 0 \quad \Rightarrow$$

$$\boxed{\ddot\theta + \tfrac{g}{l}\sin\theta - \tfrac{a}{l}\omega^2\cos(\theta-\omega t) = 0}$$