January 2017 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^8$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Hydrogen atom wave functions: $\Phi_{nlm}(r,\theta,\varphi) = R_{nl}(r)Y_{lm}(\theta,\varphi)$
$\Phi_{100}(r,\theta,\varphi) = \pi^{-1/2}\ a_0^{-3/2} \exp(-r/a_0)$,
$\Phi_{200}(r,\theta,\varphi) = (4\pi)^{-1/2}\ (2a_0)^{-3/2}\ (2 - r/a_0) \exp(-r/(2a_0))$
$\Phi_{211}(r,\theta,\varphi) = (8\pi)^{-1/2}\ (2a_0)^{-3/2}\ (r/a_0) \exp(-r/(2a_0)) \sin\theta\ e^{i\varphi}$
$\Phi_{210}(r,\theta,\varphi) = (4\pi)^{-1/2}\ (2a_0)^{-3/2}\ (r/a_0) \exp(-r/(2a_0)) \cos\theta$
$\Phi_{21-1}(r,\theta,\varphi) = (8\pi)^{-1/2}\ (2a_0)^{-3/2}\ (r/a_0) \exp(-r/(2a_0)) \sin\theta\ e^{-i\varphi}$

Useful integral: $\int_0^\infty e^{-br}\ r^n\ dr = n!/b^{n+1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Determine the energies and the degeneracies of the two lowest levels of a system composed of three particles with equal masses m, where the particles are

(a) distinguishable non-interacting spinless particles in a 3-dimensional simple harmonic potential with spring constants $k_x = k_y = k_z = k$.

(b) distinguishable non-interacting spinless particles in a 3-dimensional Coulomb potential $V(x,y,z) = -Ze^2/r$, where $r = (x^2 + y^2 + z^2)^{1/2}$.

(c) indistinguishable non-interacting spin ½ particles in a 3-dimensional cubic box (with impenetrable walls) of dimensions $L_x \times L_y \times L_z$, where $L_x = L_y = L_z = L$.

Problem 2:

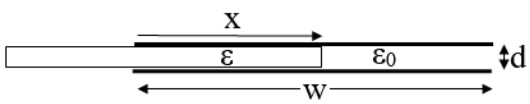
(a) A coordinate system S′ moves with constant velocity along the x axis of a second coordinate system S. Show that the space-time interval between two events $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2$ is invariant under a Lorentz transformation.

(b) If, as measured in S, two events occur at the same place and are separated in time by Δt, what is the spatial separation of these two events as measured in S′, if they are observed to be separated by a time $\Delta t'$ in the S′ frame?

(c) Two events in Minkowski space have a time-like separation. Show that the time between the events as measured by any inertial observer is always greater than or equal to the proper time between the events.

(d) The proper time interval between the same two events in Minkowski space is measured by two different observers. Is it possible that their measurements do not agree within experimental uncertainty? If yes, give an example.

Problem 3:
A parallel plate capacitor has square plates of width w and plate separation d.
A square dielectric also of width w and thickness d, with permittivity ε and mass m, is inserted between the plates a distance x into the capacitor and held there.
The plates are connected to a battery with battery voltage V_0.



(a) Derive a formula for the force exerted on the dielectric as a function of x.
Neglect edge effects.

Assume that the battery stays connected and the dielectric is released from rest at x = w/2.
Describe its subsequent motion. What is the range of the dielectric's motion, and within that range, what is the dielectric's speed as a function of position x?
(Neglect friction, ohmic heating, radiation.)

What is the maximum speed v_{max} of the dielectric?
Express your answers in terms of ε_0, ε, V_0, w, d and m.

(b) Now assume that the dielectric is released from rest at x = w/2 after the battery has been disconnected.
Derive a formula for the force exerted on the dielectric for w/2 < x < w. Neglect edge effects.

What is the maximum speed of the dielectric?
Express your answer in terms of ε_0, ε, V_0, w, d and m.

(c) Find the ratio v_{max} (case a) to v_{max} (case b) in terms of ε_0 and ε.

Problem 4:
Particles of mass m are scattered by a spherically symmetric potential energy function

$U(r) = \alpha\delta(r - a)$,

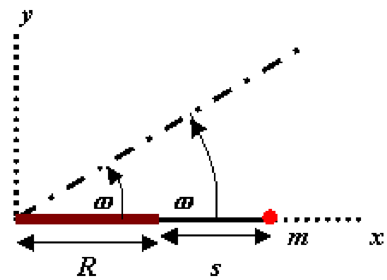
where α and a are constants and α is small.
(a) Calculate the scattering amplitude in the Born approximation.
(b) Calculate the scattering amplitude, the differential scattering cross section, and the total cross section in the low-energy Born approximation.

Problem 5:

A massless rod of length R is caused to rotate about one end with constant angular frequency ω on a frictionless horizontal table in the x-y plane. A massless string of length s is tied to the other end of the rod, and a point mass m is attached to the far end of the string.

I. At time $t = 0$, both the rod and the string lie on the x-axis, and m is given a velocity $\omega(R + s)$ in the +y- direction.

A. Which of the following are conserved in the motion that follows, and why?
   (a) Linear momentum **p**
   (b) Energy E
   (c) Angular momentum **L**



B. Find the trajectory as a function of time.

II. Suppose that the mass is given an initial velocity that is in the y-direction, but slightly different from $\omega(R + s)$ in magnitude. Show that the mass will execute simple harmonic motion about a line, which is an extension of the rod. Find the frequency of the oscillation. Use the small angle approximation freely.

Problem 6:

A simple harmonic oscillator of mass m and resonant frequency ω is restricted to move in one dimension. Let x denote the displacement from equilibrium. A time-dependent force F(t) of finite duration acts on the oscillator,

(a) Show that the equation of motion can be written as $dz/dt + i\omega z = F(t)/m$, where $z = dx/dt - i\omega x$.

(b) If the oscillator is initially at rest at equilibrium, show that the energy transferred to the oscillator in the limit $t \rightarrow \infty$ can be written as
$E(t \rightarrow \infty) = (2m)^{-1} \left|\int_0^\infty F(t) \exp(i\omega t)\, dt\right|^2$.

(c) For the force $F = F_0$ for $0 < t < t_0$ and $F = 0$ for all other times, where F_0 is a constant, find the duration t_0 of the force that transfers maximum energy to the oscillator.

Problem 7:
A hydrogen atom with Hamiltonian $H_0(r)$ is placed in a time-dependent electric field $\mathbf{E} = E(t)\,\mathbf{k}$.
The perturbed Hamiltonian is $H(\mathbf{r},t) = H_0(r) + H'(\mathbf{r},t)$.
(a) Show that $H'(\mathbf{r},t) = q_e E(t)\, r \cos(\theta)$.

(b) Assuming the electron is initially in the ground state, and recalling that the first excited state of hydrogen is quadruply degenerate, to which state of the quadruply degenerate first excited states is a dipole transition from the ground state possible? Prove this.

(c) If the electron is in the ground state at $t = 0$, find the probability (to first order in perturbation theory) that at time t the electron will have made the transition to the state determined in (b), as a function of $E(t)$.

Problem 8:
(a) Write down Maxwell's equations for a conducting medium with conductivity σ and permittivity ε_0.

(b) A plane wave of low frequency $\omega << \sigma/\varepsilon_0$ is propagating in the z-direction inside the conducting medium.
Let $\mathbf{E} = \mathbf{E}_0\exp(i(kz - \omega t))$, $\mathbf{B} = \mathbf{B}_0\exp(i(kz - \omega t))$, where $\mathbf{E}_0$, $\mathbf{B}_0$, and k are complex.
Use Maxwell's equations to show that $k = k_1 + ik_2$, $k_1 \approx k_2 \approx (\mu_0\omega\sigma/2)^{1/2}$.
Calculate the ratio of the complex amplitude of the two fields, E_0/B_0 (magnitude and phase).

(c) Calculate the energy flux (time averaged Poynting vector) in the conducting medium.