Fall 2010 Qualifying Exam

Part II

Mathematical tables are provided. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

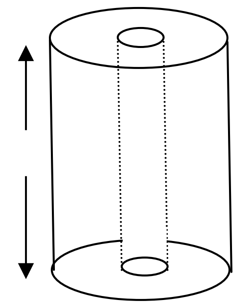
Physical Constants:

Planck constant: $h = 6.62606896 \times 10^{-34}$ Js, $\hbar = 1.05457266 \times 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 \times 10^{-23}$ J/K
Elementary charge: $e = 1.60216487 \times 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 \times 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 \times 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 \times 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 \times 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 \times 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 \times 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e\, c) = 2.42631 \times 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 \times 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 \times 10^{-8}$ W m^{-2} K^{-4}
Wien wavelength displacement law constant: $\sigma_w = 2.897\,7685 \times 10^{-3}$ mK

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

A cylindrical capacitor of length L, with an inner radius a and outer radius b, is filled with a solid dielectric (permittivity ε). If we can ignore the top/bottom end effects,



(a) find the electric field for $a < r < b$ when the charge on the capacitor is Q;
(b) find the capacitance;
(c) A potential difference V is maintained between the two cylinders. The solid dielectric is displaced down by a distance $x < L$. Find the total capacitance now, and
(d) find the magnitude and direction of the force acting on the solid dielectric.

Problem 2:
A simple pendulum of mass m_2 and length l is constrained to move in a single plane. The point of support is attached to a mass m_1 which can move on a horizontal line in the same plane.
(a) Find the Lagrangian of the system in terms of suitable generalized coordinates.
(b) Derive the equations of motion.
(c) Find the frequency of small oscillations of the pendulum.

Problem 3:
An object on a planar platform moves in an elliptical trajectory described by

$$x(t) = x_1 + x_2 \cos(\alpha t), \quad y(t) = y_1 + y_2 \sin(\alpha t),$$

where x and y are measured with respect to a coordinate system fixed on the platform. The platform is rotating with respect to an inertial coordinate system XYZ. The two coordinate systems XYZ and xyz are the same at time t = 0. The axis of rotation is the Z-axis, but the angular speed of rotation is fluctuating with time and is given by

$$\omega(t) = \omega_1 + \omega_2 \sin(\beta t).$$

Find an expression for the velocity components V_x and V_y of the object in the inertial frame at time t.

Problem 4:

A one-dimensional harmonic oscillator has mass m and angular frequency ω. Denoting the momentum by p and the coordinate by x, we can define the operators

$$a = \alpha x + i\beta p, \quad a^\dagger = \alpha x - i\beta p,$$

where $\alpha = \sqrt{\frac{m\omega}{2\hbar}}, \quad \beta = \sqrt{\frac{1}{2m\omega\hbar}}$.

(a) Find $[a, a^\dagger]$.

(b) Find the Hamiltonian in terms of a and $a^\dagger$.

(c) These operators, when acting over the eigenstates of the harmonic oscillator with quantum number n, denoted by |n>, have the property
$a^\dagger|n> = \sqrt{n+1}|n+1>, \ a|n> = \sqrt{n}|n-1>$.
Find the ground state expectation value of x^4.
Show your work!

Problem 5:

Inside a blackbody cavity, the energy density per unit frequency interval, ρ(ν), is given by Planck's formula

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}.$$

(a) Derive an expression for the intensity per unit frequency interval, I(ν), of the radiation emitted by the blackbody.
(b) Derive the Stefan-Boltzman law.
(c) Derive Wien's displacement law.

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

Problem 6:

Consider a one-dimensional quantum-mechanical scattering problem, involving a particle of mass m moving through a region with potential energy function

$$V(x) = V_0, \ 0 \le x \le L, \quad V(x) = 0 \ \text{otherwise}.$$

The particle moves from -∞ to +∞. Assume that its energy is chosen to be exactly V_0.
Find the transmission and the reflection probabilities.

Problem 7:

A pair of parallel conducting rails a distance d apart is placed in a uniform magnetic field $\vec{B}$ which is perpendicular to the rails. A resistance R is connected across the rails and a conducting bar of mass m and negligible resistance is placed at rest on the rails and perpendicular to them. A constant force F is applied to the bar pulling it along the rails.

(a) What is the value of v when the bar's acceleration becomes zero?

(b) Derive an expression for the speed v(t) of the bar as a function of time.

(c) If F is suddenly reduced to zero at time $t' = \ln(2)\, mR/(Bd)^2$, find the rate of decrease of the kinetic energy of the bar for $t > t'$.

(d) Show that the rate of decrease of the kinetic energy of the bar is equal to the ohmic heating rate.

Problem 8:

The Hamiltonian matrix for a two-state system can be written as

$$H = \begin{pmatrix} E_1^0 & \lambda\Delta \\ \lambda\Delta & E_2^0 \end{pmatrix} .$$

Clearly the eigenfunctions for the unperturbed problem ($\lambda = 0$) are given by

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Solve this problem exactly to find the eigenvalues E_1 and E_2 of H.

Examine your expressions for E_1 and E_2 when making two different assumptions.

(i) Assume that $\lambda|\Delta| << |E_1^0 - E_2^0|$. Expand the expressions for E_1 and E_2 in powers of $(\lambda\Delta)^2/(E_1^0 - E_2^0)^2$, and keep only terms up to first order.

(ii) Assume that $E_1^0 = E_1^0 = E$ and simplify the expressions for E_1 and E_2.

(b) Assuming that $\lambda|\Delta| << |E_1^0 - E_2^0|$, solve the same problem using time-independent perturbation theory up to first order in the energy eigenfunctions and up to second order in the energy eigenvalues. Compare the eigenvalues with the results obtained in part (a).

(c) Suppose the two unperturbed energies are "almost degenerate," that is, $|E_1^0 - E_2^0| << \lambda|\Delta|$. Show that the exact results obtained in (a) closely resemble what you would expect by applying degenerate perturbation theory to this problem with E_1^0 set exactly to equal to E_2^0.