August 2016 Qualifying Exam

Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^8$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Useful integrals:
$\int x^2 \sin^2(x)\,dx = x^3/6 - (x^2/4 - \tfrac{1}{8}) \sin(2x) - \tfrac{1}{4} x \cos(2x)$
$\int x^2 \cos^2(x)\,dx = x^3/6 + (x^2/4 - \tfrac{1}{8}) \sin(2x) + \tfrac{1}{4} x \cos(2x)$

Section I:
Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

Problem 1:
A spherical, shiny holiday decoration ball is acting as a convex mirror. The sphere has a radius of 4 cm. Your eye is 10 cm from the mirror. How much bigger or smaller is the image of your eye than the actual size of your eye? Is the image real or virtual, upright or inverted?

Problem 2:
A beaker of mass 1 kg containing 2 kg of water rests on a scale. A 2 kg block of aluminum (specific gravity of 2.7) is suspended from a spring scale and is submerged in the water.
(a) What does the upper spring scale read for the weight of the aluminum block?
(b) What does the lower scale read for the weight of the whole system?

Problem 3:
Assume that an electromagnetic wave moving through vacuum has a magnetic field given by
$\mathbf{B}(\mathbf{r}, t) = B_0 \sin(-k(y + ct) + \pi/4)\, \mathbf{k}$.
(a) What is the direction of propagation?
(b) What is the associated electric field?
(c) Define the Poynting vector and give the Poynting vector for the electromagnetic wave in this problem.

Problem 4:
A board weighing 90 N and 12 m long rests on two supports with each support being one meter from its respective ends. A 360 N block is placed on the board 2 m from the rightmost support. Find the force that is exerted by each support on the board.

Problem 5:
Find an expression for the radius of the trajectory of a particle of charge q moving at a speed v at right angles to a uniform magnetic field of magnitude B, if $v \ll c$. Estimate the radius of motion for an electron and also that for a proton assuming that they move at velocity ~0.1 c at right angles to a magnetic field of 5×10^{-6} G, a typical interstellar magnetic field. Will such protons and electrons be confined to a galaxy of size 10^5 light years?
(1G = 10^{-4} T)

Problem 6:
Find the allowed energies of the "half" harmonic oscillator
$U(x) = \frac{1}{2}m\omega^2x^2$, $x > 0$, $U(x) = \infty$, $x \le 0$.

Problem 7:
Calculate the energy of the electrostatic interaction between a point charge q placed in the center of a spherical cavity of radius R, which was cut inside a very large grounded conductor, and the conductor.

Problem 8:
A cylinder of mass m, radius r, and length l rolls straight down an inclined plane of length L and angle 30° with respect to the horizontal. A sphere of mass M, and radius R rolls down another inclined plane, also of length L but at an angle θ with respect to the horizontal. Each object is solid, of uniform density, and rolls without slipping. Find an expression for the angle θ for which the two objects reach the bottom at the same time after they are released from rest at the same time.

Problem 9:
An operator $\hat{A}$, representing observable A, has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. An operator $\hat{B}$, representing observable B, has two normalized eigenstates, φ_1 and φ_2, with eigenvalues b_1 and b_2, respectively. The eigenstates are related by
$\psi_1 = (3\varphi_1 + 4\varphi_2)/5$, $\psi_2 = (4\varphi_1 - 3\varphi_2)/5$.
(a) Observable A is measured, and the value a_1 is obtained. What is the state of the system immediately after this measurement?
(b) If B is measured immediately afterwards, what are the possible results, and what are their probabilities?
(c) If the result of the measurement of B is not recorded and right after the measurement of B, A is measured again, what is the probability of getting a_1?

Problem 10:
A point mass m attached to the end of a string revolves in a circle of radius R on a frictionless table at constant speed with initial kinetic energy E_0. The string passes through a hole in the center of the table and the string is pulled down until the radius of the circle is ½ of its initial value. Assuming no external torque acts on the system, how much work is done?

Section II:
Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

Problem 11:
Calculate the uncertainty product $<\Delta x><\Delta p>$ for the ground state of the infinite square-well potential. How can we tell if the ground-state wave function is or is not a minimum uncertainty state?

Problem 12:
The source of the first gravitational wave event observed by the LIGO collaboration in 2015 has been interpreted as the merger of two black holes in a binary system, each with a mass of roughly 35 solar masses (implying a radius for the event horizon of about 100 km for each, if assumed spherical), where a solar mass is $1.989 * 10^{30}$ kg. A full understanding requires general relativity, but assume Newtonian mechanics and Newtonian gravity as a first approximation for the orbital motion. At the peak amplitude of the detected gravitational wave, its measured frequency indicated that the two black holes were revolving around the center of mass about 75 times per second.
What was the approximate separation of the centers for the two black holes at this point in the merger event?

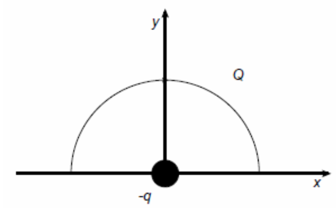
Problem 13:
Consider a spinless particle of mass m moving in one dimension in the presence of a delta-function potential well, $U(x) = -\lambda\, \delta(x)$, $\lambda > 0$.
(a) Evaluate the transmission coefficient $T(E)$ as a function of the incident energy $E > 0$.
(b) Find the energy of the bound state for this potential well.
(c) Comment on the pole in the expression for $T(E)$ from part (a) in light of your result from part (b).

Problem 14:

(a) A positive charge Q is spread over an semicircular arc with radius R as shown.

(1) What is the work required to bring in a charge -q from infinity to the center of the arc?

(2) Calculate the magnitude and direction of the force on a charge –q at the center of the arc.



(b) The potential of a uniformly charged spherical shell of radius R centered at the origin is
$V(r) = q/(4\pi\varepsilon_0 r)$ $r \geq R$, $V(r) = q/(4\pi\varepsilon_0 R)$ $r < R$,
where q denotes the total charge of the sphere.

Calculate the energy that it requires to deposit a charge Q on an initially neutral conducting spherical shell with radius R. Use two different approaches to come to the result.

(3) Calculate the energy by incrementally adding a charge dq to the sphere.

(4) Obtain the energy by considering the resulting electric field of the spherical shell.

Problem 15:

(a) Show that the eigenvalues of a general 2×2 matrix A can be expressed as

$$\lambda_\pm = \frac{1}{2}\mathrm{T} \pm \sqrt{\frac{1}{4}\mathrm{T}^2 - \mathrm{D}},$$

where D is the determinant of A and T is the trace of A (sum of diagonal elements).

Show that the matrix $\mathrm{M} = \begin{pmatrix} 0 & \mathrm{m} \\ \mathrm{m} & \mathrm{M} \end{pmatrix}$,

with M >> m has two eigenvalues, with one much larger than the other.

(b) Show that the most general form of a 2×2 unitary matrix U with unit determinant can be parameterized as $\mathrm{U} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ -\mathrm{b}^* & \mathrm{a}^* \end{pmatrix}$,

subject to the constraint $aa^* + bb^* = 1$, where * denotes complex conjugation.

Thus, it can be parameterized by a single (complex) parameter.