January 2024 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\lambda)) - 1]^{-1}$

Trigonometric identity:
$\cos(a) - \cos(b) = -2\sin((a - b)/2)\sin((a + b)/2)$.

Useful integral:
$\int_0^\infty x^{2n} \exp(-x^2)dx = [(1 * 3 * 5 * ... * (2n - 1))/2^{n+1}]\pi^{1/2}$.

Solve 6 out of the 8 problems! (All problems carry the same weight)

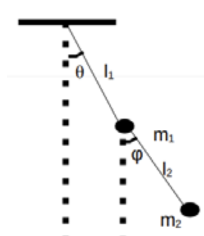
Problem 1:
Consider a potential barrier with height U and width a.
(a) Which will tunnel more easily through the potential barrier, an electron or a proton? Why? Assume each particle's kinetic energy is smaller than U.
(b) If the incoming particle's kinetic energy is 32.0 eV, U = 41 eV, and a = 0.25 nm, find the probability that the particle will tunnel through the barrier, both for the proton and the electron.

Problem 2:
A vertical spring has a spring constant k = 48 N/m. At t = 0 a force F(t) = 51 cos 4t (t ≥ 0) (SI units), initially in the downward direction, is applied to a 30 N weight which hangs in equilibrium at the end of the spring.
(a) Neglecting damping, find the position of the weight at any time t. Let g = 9.8 m/s^2.
(b) Neglecting damping, find the position of the weight at any time t if you let g = 10 m/s^2.

Problem 3:
Consider the one-dimensional harmonic oscillator with potential $V(x) = m\omega^2x^2/2$. Use the variational principle to find the ground state energy employing a Gaussian wave function
$\psi(x) = (b/\pi)^{1/4}\exp(-bx^2/2)$,
which is already normalized to 1. "b" is a variational parameter that you must optimize.
(a) Find the kinetic energy $\langle T \rangle$ in this state.
(b) Find the potential energy $\langle V \rangle$ in this state.
(c) Optimize "b".
(d) Find the optimal variational energy. How does the result compare with the exact result for a 1D harmonic oscillator?

Problem 4:
A particle of charge q and mass m moves in a region containing a uniform electric field
E = E**i** pointing in the x-direction and a uniform magnetic field **B** = B**k** pointing in the z-direction.
(a) Write down the equation of motion for the particle and find the general solutions for the Cartesian velocity components $v_i(t)$ in terms of B, E, q, and m.
(b) Solve for the position of the particle as a function of time if the particle is released from rest at t = 0.

Problem 5:
(a) For the system shown, consider only motion in the vertical plane.
Write the Lagrangian for small displacements from equilibrium of the system in the form



$$L = \frac{1}{2}[\dot{\theta} \quad \dot{\phi}]T\begin{bmatrix}\dot{\theta}\\ \dot{\phi}\end{bmatrix} - \frac{1}{2}[\theta \quad \phi]K\begin{bmatrix}\theta\\ \phi\end{bmatrix},$$

where T and K are matrices. Clearly identify (write out) T and K.
If $l_1 = l_2 = l$, and $m_1 = 21$ and $m_2 = 4$ (everything in SI units), find the frequencies of the normal modes of this system? Briefly describe these modes.

Problem 6:

A thermodynamic system consists of an ideal monatomic gas confined in a cylinder by a frictionless piston. What physical process would have to be carried out to change the temperature at constant entropy S_1 from an initial value T_1 to a greater final value T_2 and what physical conditions would have to be satisfied by the all of the container?

Give a quantitative expression for the physical change in terms of T_1 and T_2.

Problem 7:

A particle of mass m in an infinite square well of width L starts out in the left half of the well and is at t = 0 equally likely to be found at any point in that region. Assume that at t = 0 the particle's wave function is constant for $0 < x < L/2$, and zero everywhere else.

(a) What is its initial normalized state $\psi(x,0)$?

(b) Expand $\psi(x,0)$ in terms of the eigenfunctions of the Hamiltonian for the infinite square well and determine the expansion coefficients.

(c) What is the probability that a measurement of the energy at t = 0 would yield the value $\pi^2\hbar^2/(2mL^2)$ in that state?

(d) Assume that the energy in (b) is measured at time t_m.

Determine the state of the particle for all $t > t_m$.

Problem 8:

Prior to the discovery of nuclear reactions, the leading suggestion for the source of the solar luminosity was gravitational contraction. Estimates of the age of the sun were made given its mass $M_{sun} = 1.99*10^{30}$ kg, its radius $R_{sun} = 6.96*10^5$ km, and the current solar luminosity $L_{Sun} = 3.94 \times 10^{26}$ W .

For a star in mechanical equilibrium the inward pressure due to gravity must be balanced by the outward pressure due to its internal kinetic energy. A very simple model assumes the sun is a hot spherical ball of hydrogen with uniform number density n = N/V. For such a star of mass M and radius R the pressure as a function of the radial distance r from the center is given by

$P(r) = 3GM^2(R^2 – r^2)/(8\pi R^6)$.

Using the ideal gas law, $P(r) = nkT(r)$, and relating the internal energy density $u_i(r)$ to the temperature $T(r)$ for a monatomic gas, we have $u_i(r) = (3/2)P(r)$.

(a) Find the total internal energy U_i stored in the star as a function of M and R.

(b) Find the gravitational potential energy U_g of the star as a function of M and R, assuming that the potential is zero at infinity.

(c) If we assume that today's sun has collapsed from a very large size, how much of gravitational potential energy has been radiated away.

(d) Assuming the luminosity of the sun has been roughly constant and gravitational contraction is the source of all its radiated energy, estimate the age of the sun.