January 2023 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

$$\int \frac{x}{\sqrt{a-bx}} = -\frac{(a+bx/2)}{3b^2}\sqrt{a-bx}$$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Joule performed the following experiment on a gas of N ideal molecules contained in a thermally isolated box with initial temperature T.

-- The gas was initially confined in a volume V_1 that was separated from another vacant volume V_2 by a partition.

-- The partition was suddenly removed and the gas underwent free expansion to a larger volume $V_1 + V_2$, while the whole system was kept thermally isolated.

-- It was found that the final temperature T_f was equal to the initial temperature T. This was found true for any T, V_1, and V_2.

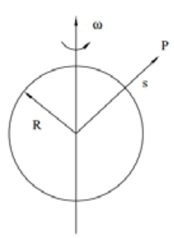
(a) From this experiment, what conclusion can we draw regarding the internal energy of the ideal gas U(T, V) as a function of temperature T and volume V?

(b) With the same initial condition as above, we now let the system be in thermal contact with a reservoir at temperature T, while we slowly move the partition to have the same final volume $V_1 + V_2$. Find the change of the entropy of the ideal gas between the initial and final states. Assume the ideal gas equation of state with the Boltzmann constant k.

(c) Suppose that the heat capacity at constant volume of the system is found to be $C_V = NkT^{\gamma}$ with a constant $\gamma > 0$. With the same initial condition as above, we now move the partition slowly to the same final volume, while keeping the system thermally isolated. Find the final temperature of the system.

Problem 2:

A spherical shell of radius R carrying a uniform surface charge σ is set spinning at angular velocity ω.



(a) Find the vector potential **A** that it produces at point P located at a distance s, s > R from the center of the sphere.

(b) Find the magnetic field **B** outside the sphere.

(c) How does the result for **A** change if r < R?

(d) Find **B** inside the sphere.

Problem 3:

Assume that the Hamiltonian for a spin ½ particle in a magnetic field **B** is given by $H = g\, \boldsymbol{\sigma}\cdot\mathbf{B}$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector for Pauli matrices and g is a positive real constant with the appropriate units. The magnetic field can be written as $\mathbf{B} = B_0(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$, with $0 \le \theta < \pi$, $0 \le \phi < 2\pi$.

(a) Find eigenstates and the corresponding eigenvalues of H.

(b) Let us denote the eigenstate with a negative eigenvalue by $|n-\rangle$.
Calculate $\mathbf{A} = (A_\theta, A_\phi)$, where $A_\alpha = i\langle n-|\partial/\partial\alpha|n-\rangle$, for $\alpha = \theta, \phi$.
Calculate $\Omega_{\theta,\phi} = \partial A_\phi/\partial\theta - \partial A_\theta/\partial\phi$.

(c) Are **A** and $\Omega_{\theta,\phi}$ invariant under a local gauge transformation $|n-\rangle \rightarrow e^{i\phi}|n-\rangle$?

Problem 4:

You have a system of 4 particles of equal mass M = 1 kg. One sits at the origin (x, y, z) = (0, 0, 0) m. The others sit at (1, 1, 0), (0, 1, 1) m, (1, 0, 1) m. These particles are attached to each other by massless rigid rods such that they will always maintain their positions relative to one another.

(a) Calculate the full moment of inertia tensor for this system.

(b) Find the principal axes for this system.

In case it's useful to you, here's a fact:
$(4 - x)^3 - 3(4 - x) - 2 = (2 - x)(5 - x)^2$.

Problem 5:

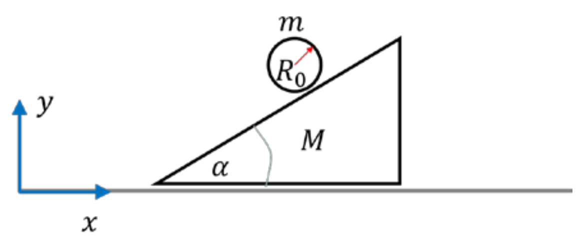
Consider the one-dimensional time-independent Schroedinger equation for some arbitrary potential V(x). Prove that if a solution $\psi(x)$ has the property that $\psi(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, then the solution must be non-degenerate and therefore real, apart from a possible overall phase factor.
Hint: Show that the contrary assumption leads to a contradiction.

Problem 6:

A uniform-density round ball with a radius R_0 and mass m is placed on a wedge of mass M and opening angle α (see below) and is subjected to a gravitational force mg. Friction between the wedge and the horizontal surface is negligible.

(a) Consider a case when there is also no friction between the wedge and the ball, and the ball slides down the wedge without any rotation. Find the acceleration **vector** of the wedge and the ball with respect to surface.

(b) Consider now a case when there is friction between the wedge and the ball and the ball rolls down the slope without sliding. What is the acceleration of the wedge now (no need to calculate the one for the ball)? How does the acceleration of the wedge compare to the acceleration found in part (a) above (is it smaller, larger or the same)?



Problem 7:

A horizontal one-meter-long stick is moving with velocity u **j** along the y−direction in the lab frame S. Reference frame S' is moving with velocity **v** = v **i** along the x-direction relative to S.

(a) Using the Lorentz transformation, derive the velocity **u**' of the stick measured by an observer at rest in S' in terms of u, v, and c.

(b) In the S frame the stick is oriented horizontally. Assume that the stick is crossing the x-axis at t = 0, with its left and right ends at $x_L = -0.5$ m, $y_L = 0$ and $x_R = +0.5$ m, $y_R = 0$ respectively.
The rod is not horizontal for observers in the S' frame. Find the vertical distance that the rod travels while crossing the x'-axis according to an observer at rest in S', in terms of u, v and c.

(c) Find the length of the stick and the angle θ that the stick makes with the x'-axis in terms of u, v, and c according to the observer in the S' frame.

Problem 8:

An insulated conducting sphere of radius R is held fixed, with its center at the origin. It carries a net charge +q. A positive charge +10q is placed on the z-axis at z = 2R.

(a) Find the force (magnitude and direction) acting on the +10q charge in units of $q^2/(4\pi\varepsilon_0 R^2)$.

(b) If the +10q charge is placed on the z-axis at z = 4R, find the force (magnitude and direction) acting on the +10q charge.

(c) Can the +10q charge be placed on the z-axis at a finite position z, so that the net force is zero? Explain your answer.