

# January 2023 Qualifying Exam

## Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

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### Physical Constants:

**Planck constant:**  $h = 6.62606896 * 10^{-34}$  Js,  $\hbar = 1.054571628 * 10^{-34}$  Js

**Boltzmann constant:**  $k_B = 1.3806504 * 10^{-23}$  J/K

**Elementary charge:**  $q_e = 1.602176487 * 10^{-19}$  C

**Avogadro number:**  $N_A = 6.02214179 * 10^{23}$  particles/mol

**Speed of light:**  $c = 2.99792458 * 10^8$  m/s

**Electron rest mass:**  $m_e = 9.10938215 * 10^{-31}$  kg

**Proton rest mass:**  $m_p = 1.672621637 * 10^{-27}$  kg

**Neutron rest mass:**  $m_n = 1.674927211 * 10^{-27}$  kg

**Bohr radius:**  $a_0 = 5.2917720859 * 10^{-11}$  m

**Compton wavelength of the electron:**  $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$  m

**Permeability of free space:**  $\mu_0 = 4\pi * 10^{-7}$  N/A<sup>2</sup>

**Permittivity of free space:**  $\epsilon_0 = 1/\mu_0 c^2$

**Gravitational constant:**  $G = 6.67428 * 10^{-11}$  m<sup>3</sup>/(kg s<sup>2</sup>)

**Stefan-Boltzmann constant:**  $\sigma = 5.670 400 * 10^{-8}$  W m<sup>-2</sup> K<sup>-4</sup>

**Wien displacement law constant:**  $\sigma_w = 2.897 7685 * 10^{-3}$  m K

**Planck radiation law:**  $I(\lambda, T) = (2hc^2/\lambda^5)[\exp(hc/(kT \lambda)) - 1]^{-1}$

### Spherical harmonics

$Y_{00} = (4\pi)^{-1/2}$ ,  $Y_{1\pm 1} = \mp (3/8\pi)^{1/2} \sin\theta \exp(\pm i\varphi)$ ,  $Y_{10} = (3/4\pi)^{1/2} \cos\theta$ ,

$Y_{2\pm 2} = (15/32\pi)^{1/2} \sin^2\theta \exp(\pm i2\varphi)$ ,  $Y_{2\pm 1} = \mp (15/8\pi)^{1/2} \sin\theta \cos\theta \exp(\pm i\varphi)$ ,

$Y_{20} = (5/16\pi)^{1/2} (3\cos^2\theta - 1)$ .

## Section I:

Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

### Problem 1:

A solid conducting sphere of radius 2.00 cm has a charge of 8.00  $\mu\text{C}$ . A conducting spherical shell of inner radius 4.00 cm and outer radius 5.00 cm is concentric with the solid sphere and has a charge of  $-4.00 \mu\text{C}$ . Find the electric field at

- $r = 1.00 \text{ cm}$ ,
- $r = 3.00 \text{ cm}$ ,
- $r = 4.50 \text{ cm}$ ,
- $r = 7.00 \text{ cm}$ ,

from the center of this charge configuration.

### Problem 2:

Complete the decays by replacing 'X' (and 'A', 'Z', 'Y', where relevant), and state the name of the decay mode

- ${}^{209}_{90}\text{Th} \rightarrow {}^A_Z\text{X} + {}^4_2\text{He}$
- ${}^{191}_{80}\text{Hg} + e^- \rightarrow {}^{191}_{79}\text{Au} + \text{X}$
- ${}^{22}_{11}\text{Na} \rightarrow {}^{22}_{10}\text{Ne} + \text{X} + \text{Y}$
- ${}^{137}_{55}\text{Cs} \rightarrow {}^{137}_{56}\text{Ba} + \text{X} + \text{Y}$

**Periodic Table of the Elements**

Atomic Number  
**Symbol**  
 Name  
 Atomic Mass

1 1A 1A H Hydrogen 1.008	2 2A 2A He Helium 4.003	3 3A 3A Li Lithium 6.941	4 4A 4A Be Beryllium 9.012	5 5A 5A B Boron 10.811	6 6A 6A C Carbon 12.011	7 7A 7A N Nitrogen 14.007	8 8A 8A O Oxygen 15.999	9 9A 9A F Fluorine 18.998	10 10A 10A Ne Neon 20.180	11 11A 11A Na Sodium 22.990	12 12A 12A Mg Magnesium 24.305	13 13A 13A Al Aluminum 26.982	14 14A 14A Si Silicon 28.086	15 15A 15A P Phosphorus 30.974	16 16A 16A S Sulfur 32.066	17 17A 17A Cl Chlorine 35.453	18 18A 18A Ar Argon 39.948	19 19A 19A K Potassium 39.098	20 20A 20A Ca Calcium 40.078	21 3B 3B Sc Scandium 44.956	22 4B 4B Ti Titanium 47.867	23 5B 5B V Vanadium 50.942	24 6B 6B Cr Chromium 51.996	25 7B 7B Mn Manganese 54.938	26 8 8 Fe Iron 55.845	27 9 9 Co Cobalt 58.933	28 10 10 Ni Nickel 58.693	29 11 11 Cu Copper 63.546	30 12 12 Zn Zinc 65.38	31 13 13 Ga Gallium 69.723	32 14 14 Ge Germanium 72.631	33 15 15 As Arsenic 74.922	34 16 16 Se Selenium 78.972	35 17 17 Br Bromine 79.904	36 18 18 Kr Krypton 83.798	37 19 19 Rb Rubidium 85.468	38 20 20 Sr Strontium 87.62	39 3B 3B Y Yttrium 88.906	40 4B 4B Zr Zirconium 91.224	41 5B 5B Nb Niobium 92.906	42 6B 6B Mo Molybdenum 95.95	43 7B 7B Tc Technetium 98.907	44 8 8 Ru Ruthenium 101.07	45 9 9 Rh Rhodium 102.906	46 10 10 Pd Palladium 106.42	47 11 11 Ag Silver 107.868	48 12 12 Cd Cadmium 112.411	49 13 13 In Indium 114.818	50 14 14 Sn Tin 118.711	51 15 15 Sb Antimony 121.760	52 16 16 Te Tellurium 127.6	53 17 17 I Iodine 126.904	54 18 18 Xe Xenon 131.294	55 21 21 Cs Cesium 132.905	56 22 22 Ba Barium 137.328	57-71 Lanthanide Series	72 3B 3B Hf Hafnium 178.49	73 4B 4B Ta Tantalum 180.948	74 5B 5B W Tungsten 183.84	75 6B 6B Re Rhenium 186.207	76 7 7 Os Osmium 190.23	77 8 8 Ir Iridium 192.221	78 9 9 Pt Platinum 195.085	79 10 10 Au Gold 196.967	80 11 11 Hg Mercury 200.592	81 12 12 Tl Thallium 204.383	82 13 13 Pb Lead 207.2	83 14 14 Bi Bismuth 208.980	84 15 15 Po Polonium [209]	85 16 16 At Astatine [209]	86 17 17 Rn Radon 222.018	87 18 18 Fr Francium 223.020	88 19 19 Ra Radium 226.025	89-103 Actinide Series	104 10 10 Rf Rutherfordium [261]	105 11 11 Db Dubnium [262]	106 12 12 Sg Seaborgium [266]	107 13 13 Bh Bohrium [264]	108 14 14 Hs Hassium [265]	109 15 15 Mt Meitnerium [268]	110 16 16 Ds Darmstadtium [281]	111 17 17 Rg Roentgenium [280]	112 18 18 Cn Copernicium [285]	113 13 13 Nh Nihonium [284]	114 14 14 Fl Flerovium [289]	115 15 15 Mc Moscovium [289]	116 16 16 Lv Livermorium [293]	117 17 17 Ts Tennessine [294]	118 18 18 Og Oganesson [294]
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Alkali Metal   Alkaline Earth   Transition Metal   Basic Metal   Semimetal   Nonmetal   Halogen   Noble Gas   Lanthanide   Actinide

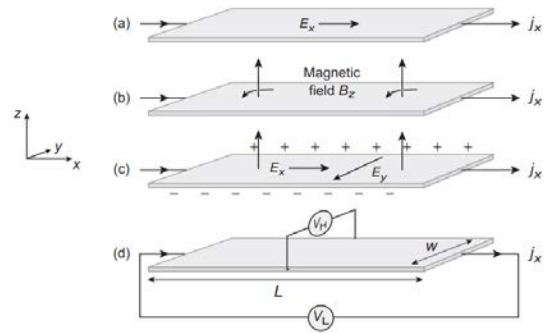
### Problem 3:

- (a) In the days before Quantum Mechanics, a big theoretical problem was to “stop” an atom from emitting light. Explain.
- (b) After Quantum Mechanics, a big theoretical problem was to make atoms in excited states emit light. Explain. What does make excited atoms emit light?

### Problem 4:

Consider the classical Hall effect experiment described by the sequence of figures.

- (a) An electric field  $E_x$  causes a current density  $j_x$  in a thin rectangular sample.
- (b) A uniform magnetic field  $B_z$  is placed in the positive  $z$  direction and the electrons respond to this field.
- (c) Electrons accumulate on one edge and a positive ion excess accumulates on the other edge, producing a transverse electric field  $E_y$  (Hall field) that just cancels the force produced by the magnetic field, so that in equilibrium current flows only in the  $x$  direction.
- (d) In a typical experiment the longitudinal voltage  $V_L$  and the transverse (Hall) voltage  $V_H$  are measured.



Find the classical electric field  $E_y$  required to cancel the effect of the magnetic field on the electrons so that no current flows in the  $y$  direction.

### Problem 5:

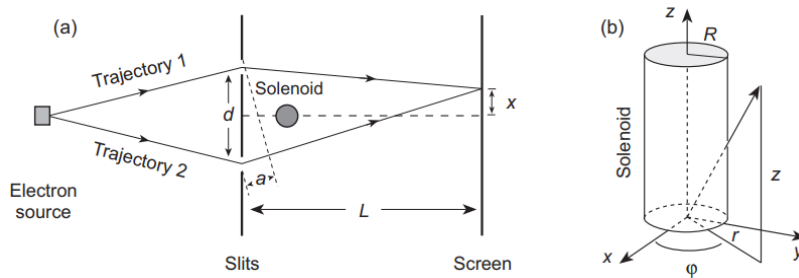
- Consider a Hydrogen atom  $m = 1.7 \cdot 10^{-27}$  kg falling from the interstellar medium onto the surface of a neutron star which has a mass equal to 1.4 times the mass of the sun,  $m_{\text{sun}} \approx 2.8 \cdot 10^{30}$  kg, and a radius of 10 km.
- (a) What is the potential energy lost by the Hydrogen atom in MeV (assuming Newtonian gravity and ignoring general relativistic effects)?
- (b) How does that compare with the rest mass energy of the Hydrogen atom?
- (c) Assume that the neutron star accretes 1 solar mass in this way every  $10^{11}$  years, and that this is the dominant source of energy for the neutron star. What is the associated neutron star luminosity (W)?

### Problem 6:

- Muons decay with a half-life of  $t_{1/2} = 1.56 \mu\text{s}$ . They are produced with very high speeds in the upper atmosphere when cosmic rays collide with air molecules. Assume the height of the atmosphere of Earth is 200 km in the reference frame of the earth.
- (a) If the muons are traveling at  $0.99998c$ , how long does it take for the muons to pass through Earth's atmosphere as viewed from the Earth?
- (b) How many half-lives is this?
- (c) How long does it take as viewed from the rest frame of the muons?
- (d) What is the height of the atmosphere in the rest frame of the muons?

**Problem 7:**

The Aharonov–Bohm experiment is illustrated in the figure below.



It is a two-slit electron scattering experiment where a solenoid is placed in the region behind the screen and between the two classical paths that electrons passing through the slits would follow to reach a point on the screen. The long and thin solenoid confines the magnetic field to regions that the electrons should not pass through. In terms of the cylindrical coordinates in Fig (b), the magnetic field may be assumed given by

Inside solenoid:  $B_r = 0, B_\phi = 0, B_z = B,$   
 Outside solenoid:  $B_r = 0, B_\phi = 0, B_z = 0.$

(a) Show that a vector potential given in the cylindrical coordinates by

inside the solenoid:  $A_r = A_z = 0, A_\phi = Br/2,$   
 outside the solenoid:  $A_r = A_z = 0, A = BR^2/(2r),$

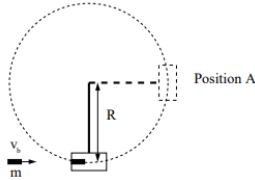
leads to the magnetic field components inside and outside the solenoid given above.

Thus, in the Aharonov–Bohm experiment the electrons never experience a finite magnetic field but they may encounter a non-zero vector potential outside the solenoid.

(b) What does this result, and that in the Aharonov–Bohm experiment the interference pattern is observed to be shifted when current is flowing in the solenoid, say about the relative importance of the magnetic field and the vector potential in classical and quantum mechanics?

**Problem 8:**

A block of mass  $2m$  is attached to a rigid massless rod of length  $R$  and is suspended from a frictionless pivot. The block is released from rest from position A with the rod extending out horizontally. When the block swings to reach the bottom of the circle, a bullet of mass  $m$  travelling at some velocity  $v_b$  strikes it as shown in the figure and lodges in the hole so that the two masses move together as one thereafter.



- Find the block's speed  $v_0$  at the bottom of the circle before being hit by the bullet.
- After the bullet hits the block, the two masses move to the right with a common speed  $2v_0$ . Find the speed of the bullet  $v_b$  just before hitting the block. Provide your answer in terms of  $v_0$ .
- Find the relative change of kinetic energy during this collision.
- Find the tension in the rod immediately after the collision while the two masses are still at the bottom of the circle. (Draw an appropriate free body diagram and plug  $v_0$  into your answer for the tension).

**Problem 9:**

Suppose an electron is in a state described by the wave function

$$\psi = (1/\sqrt{4\pi})(e^{i\phi}\sin\theta + \cos\theta)g(r),$$

where  $\int_0^\infty |g(r)|^2 r^2 dr = 1$ ,

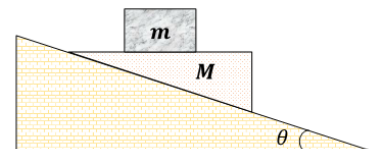
and  $\phi$ ,  $\theta$  are the azimuth and polar angles respectively.

- What are the possible results of a measurement of the z-component  $L_z$  of the angular momentum of the electron in this state?
- What is the probability of obtaining each of the possible results in part (a)?
- What is the expectation value of  $L_z$ ?

**Problem 10:**

A wedge of mass  $M$  is moving along a frictionless slope. The slope is fixed to the ground and the slope angle is  $\theta$ .

The top surface of the wedge  $M$  remains horizontal and a cube with mass  $m$  is sitting on top of the wedge. There is also no friction between the cube  $m$  and the wedge  $M$ .



- Find the relative acceleration between  $m$  and  $M$ .
- Find the magnitude of the normal force  $N$  between  $M$  and the slope.

## Section II:

Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

### Problem 11:

Consider a system of  $N$  particles with only 3 possible energy levels separated by  $\epsilon$ . Let the ground state energy be zero. The system occupies a fixed volume  $V$  and is in thermal equilibrium with a reservoir at temperature  $T$ . Ignore interactions between particles and assume Boltzmann statistics apply.

- (a) What is the partition function for a single particle in the system?
- (b) What is the average energy per particle?
- (c) What is the probability that the  $2\epsilon$  level is occupied in the high-temperature limit  $k_B T \gg \epsilon$ ?  
Explain your answer on physical grounds.
- (d) What is the average energy per particle in the high-temperature limit  $k_B T \gg \epsilon$ ?
- (e) At what energy is the ground state 1.1 times as likely to be occupied as the  $2\epsilon$  level?
- (f) Find the heat capacity  $C_V$  of the system, analyze the low- $T$  ( $k_B T \gg \epsilon$ ) and high- $T$  ( $k_B T \gg \epsilon$ ) limit, and sketch  $C_V$  as a function of  $T$ .

### Problem 12:

Find the reflection coefficient and the reflectance and transmittance for the one-dimensional potential step

$$U(x) = 0 \text{ for } x < 0,$$

$$U(x) = U_0 \text{ or } x \geq 0.$$

if the particles are incident from the right (i.e. from the  $x > 0$  region) with  $E > U_0$ .

### Problem 13:

The pressure of a non-interacting gas of bosons with mass  $m$ , chemical potential  $\mu$ , and degeneracy factor  $g$  at temperature  $T$  is

$$P = - [k_B T / (2\pi\hbar^3)] \int p^2 dp \ln(1 - \exp[(-E + \mu)/(k_B T)]),$$

$$\text{where } E = (m^2 c^4 + p^2 c^2)^{1/2}.$$

- (a) What is the restriction on the chemical potential which ensures that the pressure does not diverge?
- (b) What is the number density of bosons as a function of  $\mu$  and  $T$ ?  
(The density is the partial derivative of the pressure with respect to  $\mu$ )?
- (c) What is the number of degrees of freedom, and thus the value of  $g$ , for a photon?

**Problem 14:**

Let a space-time point be defined by the coordinates  $(x_0, x_1, x_2, x_3) = (ct, x, y, z)$ .

Two events occur at space-time points  $A = (1, 0, 0, 0)$  and  $B = (3, 3, 0, 0)$  (arbitrary length units) as measured in some frame  $S$ .

- In this frame  $S$ , which event, A or B, happens first?
- Determine if these events are space-like, time-like, or light-like separated by calculating the space-time interval between these points.
- Calculate  $A'$  and  $B'$  as seen in a new frame  $S'$ .  $S'$  moves with respect to  $S$  in the  $-x$  direction with speed  $v = 0.8c$ .
- In  $S'$ , which event happens first? Comment on any possible causal relationship between the two events.
- Draw a spacetime diagram in each frame showing the light cones from  $A'$  and  $B'$ .
- Give the coordinates for a new spacetime point  $C$  that could have been caused by event A (any will do).
- Give the coordinates for a new spacetime point  $D$  that could have caused event A (again, any will do).
- If A and B had a common cause at point  $E = (0, 0, 0, 0)$ , what is the mass of the particle that then led to B happening?

**Problem 15:**

Two wedges are placed mirror symmetrically so that the tip of one touches the tip of the other (see the figure below). The surface of each wedge is at an angle of  $\theta$  degrees relative to the ground. A small elastic ball is dropped a distance  $x$  from the tip of one of the wedges with no initial velocity. From what height  $h$  needs the ball to be dropped, so that after bouncing from the two wedges, it will reach the same height  $h$  from which it was dropped? Neglect any friction from air and consider the bouncing of the ball to be completely elastic.

