January 2023 Qualifying Exam

Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Spherical harmonics
$Y_{00} = (4\pi)^{-1/2}$, $Y_{1\pm1} = \mp(3/8\pi)^{1/2}\sin\theta\,\exp(\pm i\varphi)$, $Y_{10} = (3/4\pi)^{1/2}\cos\theta$,
$Y_{2\pm2} = (15/32\pi)^{1/2}\sin^2\theta\,\exp(\pm i2\varphi)$, $Y_{2\pm1} = \mp(15/8\pi)^{1/2}\sin\theta\cos\theta\,\exp(\pm i\varphi)$,
$Y_{20} = (5/16\pi)^{1/2}(3\cos^2\theta - 1)$.

Section I:

Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

Problem 1:

A solid conducting sphere of radius 2.00 cm has a charge of 8.00 μC. A conducting spherical shell of inner radius 4.00 cm and outer radius 5.00 cm is concentric with the solid sphere and has a charge of -4.00 μC. Find the electric field at

(a) r = 1.00 cm,

(b) r = 3.00 cm,

(c) r = 4.50 cm,

(d) r = 7.00 cm,

from the center of this charge configuration.

Problem 2:

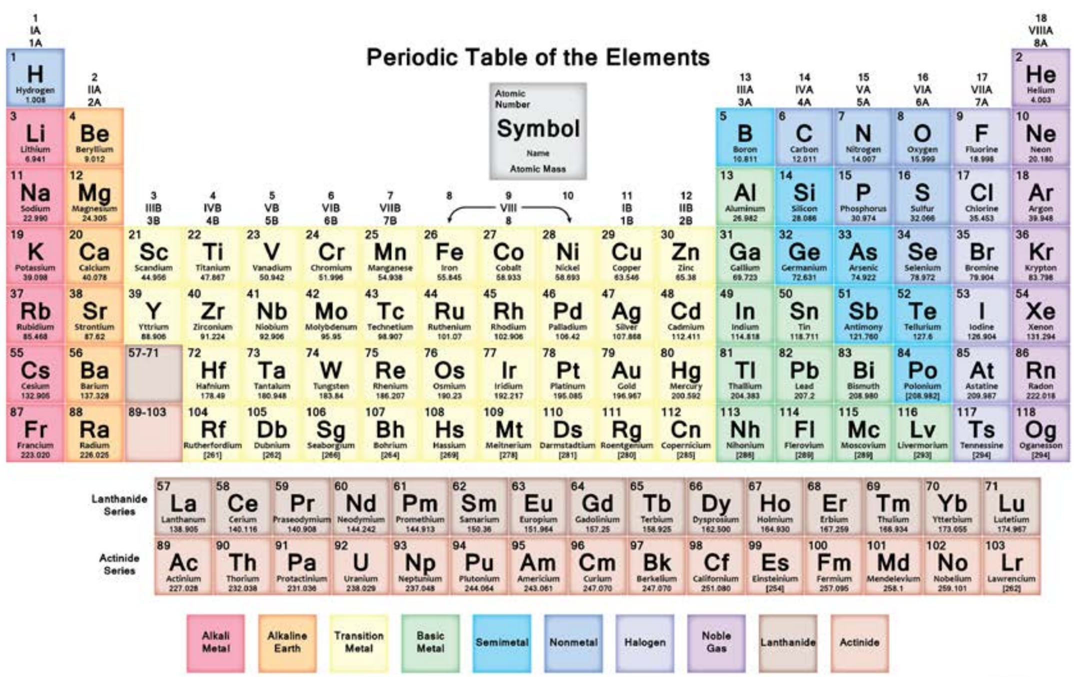
Complete the decays by replacing 'X' (and 'A', 'Z','Y', where relevant), and state the name of the decay mode

(1) ${}^{209}_{90}\mathrm{Th} \rightarrow {}^{A}_{Z}\mathrm{X} + {}^{4}_{2}\mathrm{He}$

(2) ${}^{191}_{80}\mathrm{Hg} + e^{-} \rightarrow {}^{191}_{79}\mathrm{Au} + \mathrm{X}$

(3) ${}^{22}_{11}\mathrm{Na} \rightarrow {}^{22}_{10}\mathrm{Ne} + \mathrm{X} + \mathrm{Y}$

(4) ${}^{137}_{55}\mathrm{Cs} \rightarrow {}^{137}_{56}\mathrm{Ba} + \mathrm{X} + \mathrm{Y}$



Problem 3:

(a) In the days before Quantum Mechanics, a big theoretical problem was to “stop” an atom from emitting light. Explain.

(b) After Quantum Mechanics, a big theoretical problem was to make atoms in excited states emit light. Explain. What does make excited atoms emit light?

Problem 4:

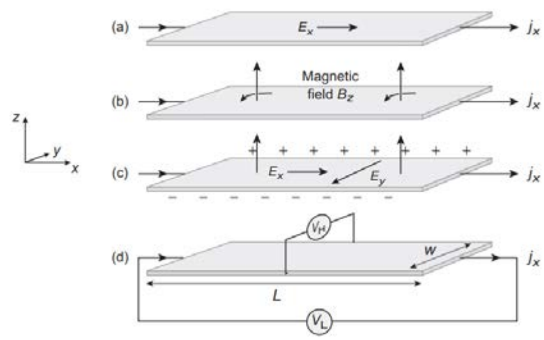
Consider the classical Hall effect experiment described by the sequence of figures.

(a) An electric field E_x causes a current density j_x in a thin rectangular sample.

(b) A uniform magnetic field B_z is placed in the positive z direction and the electrons respond to this field.

(c) Electrons accumulate on one edge and a positive ion excess accumulates on the other edge, producing a transverse electric field E_y (Hall field) that just cancels the force produced by the magnetic field, so that in equilibrium current flows only in the x direction.

(d) In a typical experiment the longitudinal voltage V_L and the transverse (Hall) voltage V_H are measured.



Find the classical electric field E_y required to cancel the effect of the magnetic field on the electrons so that no current flows in the y direction.

Problem 5:

Consider a Hydrogen atom m = $1.7*10^{-27}$ kg falling from the interstellar medium onto the surface of a neutron star which has a mass equal to 1.4 times the mass of the sun, $m_{sun} \approx 2.8 *10^{30}$ kg, and a radius of 10 km.

(a) What is the potential energy lost by the Hydrogen atom in MeV (assuming Newtonian gravity and ignoring general relativistic effects)?

(b) How does that compare with the rest mass energy of the Hydrogen atom?

(c) Assume that the neutron star accretes 1 solar mass in this way every 10^{11} years, and that this is the dominant source of energy for the neutron star. What is the associated neutron star luminosity (W)?

Problem 6:

Muons decay with a half-life of $t_{½}$ = 1.56 μs. They are produced with very high speeds in the upper atmosphere when cosmic rays collide with air molecules. Assume the height of the atmosphere of Earth is 200 km in the reference frame of the earth.

(a) If the muons are traveling at 0.99998c, how long does it take for the muons to pass through Earth’s atmosphere as viewed from the Earth?

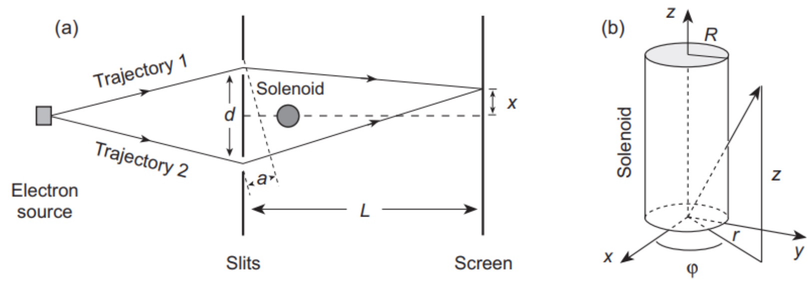
(b) How many half-lives is this?

(c) How long does it take as viewed from the rest frame of the muons?

(d) What is the height of the atmosphere in the rest frame of the muons?

Problem 7:

The Aharanov–Bohm experiment is illustrated in the figure below.



It is a two-slit electron scattering experiment where a solenoid is placed in the region behind the screen and between the two classical paths that electrons passing through the slits would follow to reach a point on the screen. The long and thin solenoid confines the magnetic field to regions that the electrons should not pass through. In terms of the cylindrical coordinates in Fig (b), the magnetic field may be assumed given by

Inside solenoid: $B_r = 0$, $B_\phi = 0$, $B_z = B$,

Outside solenoid: $B_r = 0$, $B_\phi = 0$, $B_z = 0$.

(a) Show that a vector potential given in the cylindrical coordinates by

inside the solenoid: $A_r = A_z = 0$, $A_\phi = Br/2$,

outside the solenoid: $A_r = A_z = 0$, $A = BR^2/(2r)$,

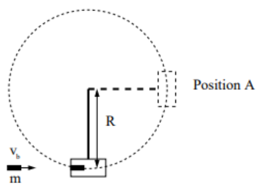
leads to the magnetic field components inside and outside the solenoid given above.

Thus, in the Aharanov–Bohm experiment the electrons never experience a finite magnetic field but they may encounter a non-zero vector potential outside the solenoid.

(b) What does this result, and that in the Aharanov–Bohm experiment the interference pattern is observed to be shifted when current is flowing in the solenoid, say about the relative importance of the magnetic field and the vector potential in classical and quantum mechanics?

Problem 8:

A block of mass 2m is attached to a rigid massless rod of length R and is suspended from a frictionless pivot. The block is released from rest from position A with the rod extending out horizontally. When the block swings to reach the bottom of the circle, a bullet of mass m travelling at some velocity v_b strikes it as shown in the figure and lodges in the hole so that the two masses move together as one thereafter.



(a) Find the block's speed v_0 at the bottom of the circle before being hit by the bullet.
(b) After the bullet hits the block, the two masses move to the right with a common speed $2v_0$. Find the speed of the bullet v_b just before hitting the block. Provide your answer in terms of v_0.
(c) Find the relative change of kinetic energy during this collision.
(d) Find the tension in the rod immediately after the collision while the two masses are still at the bottom of the circle. (Draw an appropriate free body diagram and plug v_0 into your answer for the tension).

Problem 9:

Suppose an electron is in a state described by the wave function

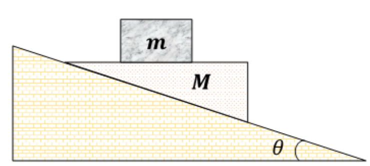
$\psi = (1/\sqrt{4\pi})(e^{i\phi}\sin\theta + \cos\theta)g(r)$,

where $\int_0^\infty |g(r)|^2 r^2 dr = 1$,
and ϕ, θ are the azimuth and polar angles respectively.

(a) What are the possible results of a measurement of the z-component L_z of the angular momentum of the electron in this state?
(b) What is the probability of obtaining each of the possible results in part (a)?
(c) What is the expectation value of L_z ?

Problem 10:

A wedge of mass M is moving along a frictionless slope. The slope is fixed to the ground and the slope angle is θ.
The top surface of the wedge M remains horizontal and a cube with mass m is sitting on top of the wedge. There is also no friction between the cube m and the wedge M.



(a) Find the relative acceleration between m and M.
(b) Find the magnitude of the normal force N between M and the slope.

Section II:

Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

Problem 11:

Consider a system of N particles with only 3 possible energy levels separated by ε. Let the ground state energy be zero. The system occupies a fixed volume V and is in thermal equilibrium with a reservoir at temperature T. Ignore interactions between particles and assume Boltzmann statistics apply.

(a) What is the partition function for a single particle in the system?
(b) What is the average energy per particle?
(c) What is the probability that the 2ε level is occupied in the high-temperature limit $k_BT >> \varepsilon$?
Explain your answer on physical grounds.
(d) What is the average energy per particle in the high-temperature limit $k_BT >> \varepsilon$?
(e) At what energy is the ground state 1.1 times as likely to be occupied as the 2ε level?
(f) Find the heat capacity C_V of the system, analyze the low-T ($k_BT >> \varepsilon$) and high-T ($k_BT >> \varepsilon$) limit, and sketch CV as a function of T.

Problem 12:

Find the reflection coefficient and the reflectance and transmittance for the one-dimensional potential step

U (x) = 0 for x < 0,

$U(x) = U_0$ or $x \geq 0$.

if the particles are incident from the right (i.e. from the x > 0 region) with $E > V_0$.

Problem 13:

The pressure of a non-interacting gas of bosons with mass m, chemical potential μ, and degeneracy factor g at temperature T is

$P = - [k_BT/(2\pi\hbar^3)] \int p^2 dp \ln(1 - \exp[(-E + \mu)/(k_BT)])$,

where $E = (m^2c^4 + p^2c^2)^{½}$.

(a) What is the restriction on the chemical potential which ensures that the pressure does not diverge?

(b) What is the number density of bosons as a function of μ and T?
(The density is the partial derivative of the pressure with respect to μ)?

(c) What is the number of degrees of freedom, and thus the value of g, for a photon?

Problem 14:

Let a space-time point be defined by the coordinates $(x_0, x_1, x_2, x_3) = (ct, x, y, z)$.

Two events occur at space-time points A = (1, 0, 0, 0) and B = (3, 3, 0, 0) (arbitrary length units) as measured in some frame S.

(a) In this frame S, which event, A or B, happens first?

(b) Determine if these events are space-like, time-like, or light-like separated by calculating the space-time interval between these points.

(c) Calculate A' and B' as seen in a new frame S'. S' moves with respect to S in the -x direction with speed v = 0.8c.

(d) In S', which event happens first? Comment on any possible causal relationship between the two events.

(e) Draw a spacetime diagram in each frame showing the light cones from A(') and B(').

(f) Give the coordinates for a new spacetime point C that could have been caused by event A (any will do).

(g) Give the coordinates for a new spacetime point D that could have caused event A (again, any will do).

(h) If A and B had a common cause at point E = (0, 0, 0, 0), what is the mass of the particle that then led to B happening?

Problem 15:

Two wedges are placed mirror symmetrically so that the tip of one touches the tip of the other (see the figure below). The surface of each wedge is at an angle of θ degrees relative to the ground. A small elastic ball is dropped a distance x from the tip of one of the wedges with no initial velocity. From what height h needs the ball to be dropped, so that after bouncing from the two wedges, it will reach the same height h from which it was dropped? Neglect any friction from air and consider the bouncing of the ball to be completely elastic.