January 2022 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

A rocket starts in frame S from rest and accelerates with 10 g as measured in its own frame. After one year, as measured in S, what will its speed v/c be with respect to S?

Problem 2:

A particle with mass m moves in one dimension ($-\infty < x < +\infty$) in an external potential that supports bound states.

When the particle is in a bound state $\psi_1(x)$, its averaged kinetic energy is E_1. $\psi_1(x)$ is a real function.

Find the averaged momentum and averaged the kinetic energy when the particle is in a state $\psi_2(x) = \psi_1(x)e^{ikx}$ (k is a real number).

Problem 3:

The surface tension γ is defined as the force along a line of unit length, $\gamma = F/L$ (N/m).

The force is parallel to the surface and perpendicular to the line.

(a) For a spherical soap bubble in equilibrium at a radius R, find the pressure difference $\Delta P = P_{in} - P_{out}$ as a function of R and γ.

(b) Assume a spherical soap bubble carries a charge Q, uniformly distributed over its surface is in equilibrium at a radius R. Find the pressure difference $\Delta P = P_{in} - P_{ouy}$ as a function of R and γ and Q.

Problem 4:

Calculate the electric quadrupole moment Q_{zz} of a uniformly charged ellipsoid of revolution about the major axis aligned with the z-axis. Let the length of the semi-major axis be b and the length of the semi-minor axis be a.

$Q_{ij} = \int(3x_i'x_j' - \delta_{ij}r'^2)\, \rho(\mathbf{r'})dV'$.

Problem 5:

Suppose we have a spin-½ particle in the quantum state $|\psi\rangle = N(e^{i\pi/4}\,|\uparrow\rangle + e^{(i\pi/5)-2}\,|\downarrow\rangle)$, with $|\uparrow\rangle$, $|\downarrow\rangle$ the states with the spin aligned parallel or anti-parallel with the z axis.

(a) Calculate the normalization constant N.

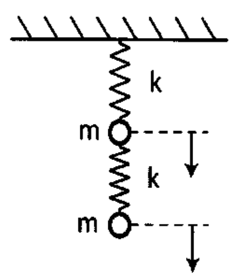
(b) Calculate the expectation value $\langle S_x\rangle$, where $\langle S_x\rangle$ is the x component of the spin angular momentum.

(c) Calculate the root-mean-square deviations ΔS_x.

Problem 6:
(a) Given two identical neutral atoms each with polarizability χ, such that the dipole moment $\mathbf{p} = \chi\varepsilon_0\mathbf{E}$, separated by a distance d, find a relation between χ and d such that the two induced dipoles are ferroelectric, i.e., parallel.
(b) Given two identical neutral atoms each with polarizability χ_e, separated by a distance d, find a relation between χ_e and d such that the dipoles are anti-ferroelectric, i.e., point in opposite directions.

Problem 7:
An FM radio station broadcasts at a frequency of f = 100 MHz using a transmitter that isotropically emits 10 kW of power.
(a) Using your knowledge of (classical) electrodynamics, find the amplitude of sinusoidal oscillation of the electric field (in V/m) and the magnetic field (in T) at a distance of 100 km from the transmitter.
(b) What is the quantum of energy carried by one photon emitted by the antenna? Express your answer three ways, in J, eV and K by using the electron charge q_e and the Boltzmann constant k_B.
(c) How many photons per second are emitted by the transmitter? How many quanta are emitted in one period of oscillation of the 100 MHz signal? [This should be a very large number, which is why the FM station is classical to a very good approximation.]
(d) An alien civilization 10 light years away from the earth is listening to the station with a noisy detector that needs to receive 100 photons to reliably say that the signal is present. Using a parabolic dish radio telescope with a diameter of 30 m, how many photons are captured per second and how long does it take to receive 100 photons?
(e) We use the station to send a data string in the form of binary numbers, e.g.
011100010101011110000111... .
The coding used is: transmitter ON = 1; transmitter OFF = 0. What would be the minimum time needed to transmit a 1 MByte = 8 Mbit digital photo of yourself to the aliens? Assume the duration of the on and off periods are equal.

Problem 8:
A particle of mass m is attached to a rigid support by a spring with a force constant k. At equilibrium, the spring hangs vertically downward. To this mass-spring combination is attached an identical oscillator, the spring of the latter being connected to the mass of the former.



(a) Show that by appropriate choice of coordinates and their zero-points the equations of motion can be expressed as
$m d^2x_1/dt^2 + 2kx_1 - kx_2 = 0,$
$m d^2x_2/dt^2 + kx_2 - kx_1 = 0.$

(b) Calculate the characteristic frequencies for one-dimensional vertical oscillations.
(c) Qualitatively describe the normal modes of the system with a short discussion plus drawings.
(d) Quantitatively compare the characteristic frequencies with the frequencies when either of the particles is held fixed while the other oscillates.