January 2020 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

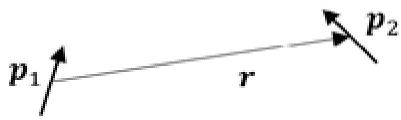
The displacement vector from electric dipole $\mathbf{p}_1$ to dipole $\mathbf{p}_2$ is $\mathbf{r}$.

(a) Calculate the electric potential energy W;

(b) Calculate the force $\mathbf{F}_{21}$ that $\mathbf{p}_1$ exerts on $\mathbf{p}_2$.

(c) Calculate the torque $\boldsymbol{\tau}_{12}$ that $\mathbf{p}_1$ exerts on $\mathbf{p}_2$.

(d) Let $\mathbf{p}_1 = (10^{-9}\ \text{Cm})\mathbf{k}$ be located at the origin and $\mathbf{p}_2 = (10^{-9}\ \text{Cm})\mathbf{i}$ at $\mathbf{r} = (3\ \text{m})\mathbf{i} + (4\ \text{m})\mathbf{k}$. Provide numeric answers for $\mathbf{F}_{21}$ and $\boldsymbol{\tau}_{12}$.



Problem 2:

Consider a spin coupled to a magnetic field via the Hamiltonian $H = \gamma \mathbf{S}\cdot\mathbf{B}$, where the three dimensional magnetic field is $\mathbf{B} = (B_x, B_y, B_z) = B_0(1, 1, 0)$, B_0 is a constant, and γ is another constant. The three dimensional spin vector is $\mathbf{S} = (S_x, S_y, S_z)$, and the S_x, S_y, S_z are the usual 2 x 2 spin matrices.

(a) In the eigenbasis of S_z, find the matrix of H and its eigenvalues.

(b) In the eigenbasis of S_z, find the two normalized eigenvectors of the H.

Problem 3:

A rocket with a proper length of 700 m is moving in the positive x-direction at a speed of 0.9c. It has two clocks, one in the nose and one in the tail, that have been synchronized in the frame of the rocket. A clock on the ground and the nose clock on the rocket both read t = 0 as they pass, i.e. have the same x-coordinate.

(a) At t = 0, what does the tail clock on the rocket read in the frame of an observer on the ground?

(b) When the tail clock on the rocket passes the ground clock,

i. what does the tail clock read in the frame of an observer on the ground?

ii. what does the nose clock read in the frame of an observer on the ground?

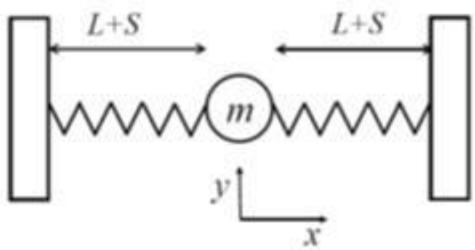
iii. what does the nose clock read in the frame of an observer on the rocket?

(c) At t = 1 h, as measured on the rocket, a light signal is sent from the nose of the rocket to an observer standing by the ground clock. What does the ground clock read when the observer receives this signal? (Assume distances perpendicular to the x-axis are negligibly small compared to distances along the x-axis.)

(d) When the observer on the ground receives the signal, he sends a return signal to the nose of the rocket. When is this signal received at the nose of the rocket as seen on the rocket?

Problem 4:

Two identical, massless springs of spring constant k and equilibrium length L are joined together and stretched each by an additional distance S. A mass m is attached to the point where the springs are joined. Note that the mass can oscillate freely in three dimensions. Neglect the effects of gravity.

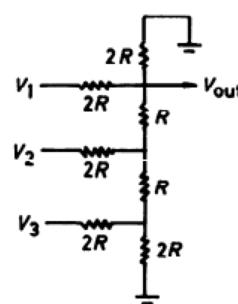


(a) For the motion along the x-axis, determine the Lagrangian of the system. Compute the Euler-Lagrange equation for x and find the angular frequency for the oscillation.
(b) Derive the angular frequency for the oscillation when the mass is displaced by a small distance in the y-direction (y-axis perpendicular to the springs).
(c) Show that for small displacements the oscillations along the x and y axes will be independent. For this find the potential energy U(x,y) - U(0,0) to second order in x and y.

Use $x,y \ll L + S$ and for small d the expansion $(1 + d)^{1/2} = 1 + \frac{1}{2}d + (1/8)d^2 + \dots$.

Problem 5: HZ

Suppose the input voltages V_1, V_2 and V_3 in the circuit shown can assume values of either 0 or 1 (0 means ground). There are thus 8 possible combinations of input voltages. List the V_{out} for each of these possibilities.



Problem 6:

You have 10 non-interacting spin ½ particles with mass M equal to the electron mass confined in a rigid 3D cube where each side has length L = 0.1 nm.

(a) There are some degeneracies in this system. Calculate the three lowest unique single-particle energy levels.

(b) Draw an energy level diagram for the ground state of the 10-particle system. Show a possible distribution of the particles among these energy levels. What is the degeneracy of the ground state of the 10-particle system?

Problem 7:

In one dimension, a particle of mass m has potential energy $U(x) = V\cos(\alpha x) - Fx$, with V a positive constant. Determine the frequency of small oscillations. What relationship between V, α, and F makes oscillatory motion possible?

Problem 8:

Consider a particle of mass m placed in an infinite two-dimensional potential well of width a.

$U(x,y) = 0$ if $0 < x < a$ and $0 < y < a$, $U(x,y) = \infty$ everywhere else.

The particle is also subject to a perturbation W described by

$W(x,y) = W_0$ for $0 < x < a/2$ and $0 < y < a/2$, $W(x,y) = 0$ everywhere else.

(a) Calculate, to first order in W_0, the perturbed energy of the ground state.

(b) Calculate, to first order in W_0, the perturbed energy of the first excited state.

Give the corresponding wave functions to 0th order in W_0.