January 2020 Qualifying Exam

Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Section I:
Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

Problem 1:
A particle of mass m and total energy $E > 0$ undergoes finite 1-D motion with potential energy $U(x) = U_0(a^2/x^2 + x^2/a^2)$, where $U_0 > 0$ and a are some parameters.
Compute the maximal speed of the particle.

Problem 2:
Find the commutator between the square of the coordinate x and the momentum p operators i.e. find $[x^2, p]$. You can use the trick of thinking of this commutator as acting over a "test function" f(x) if it helps.

Problem 3:
"Thermal" neutrons are neutrons whose energy is roughly equal to the average kinetic energy that any object has at room temperature due to thermal effects (0.04 eV).
Imagine that we prepare a beam of such neutrons.
What will be the beam's effective wavelength?

Problem 4:
Bob stands on a scale in an elevator. The elevator is accelerating upwards with an acceleration of 1 m/s^2. Bob mass is 60 kg.
(a) The scale on the elevator reads a force F'. What is F'?
(b) Once the elevator reaches its cruising speed and moves with constant velocity upward, what does the scale read now?

Problem 5:
A proton initially at rest is accelerated through a voltage of 400 V and enters a magnetic field perpendicular to its direction of travel. The magnetic field strength is 2 Tesla. What is the radius of curvature of the proton's path in the magnetic field?

Problem 6:
The Sun is located 8 kiloparsecs ($2.5*10^{20}$ meters) from the center of the Milky Way. Stars in the solar vicinity are, on average, moving in a circular orbit about the Galactic Center at a velocity of 225 kiloparsec/Gigayear (220 kilometers/second). Calculate the mass of the galaxy interior to the solar orbit.

Problem 7:
An ultra-short pulse has a duration of 8.20 fs and produces light at a wavelength of 556 nm. What is the momentum uncertainty Δp and the relative momentum uncertainty $\Delta p/p$ of a single photon in the pulse?

Problem 8:
A typical household electrical oven consists of a heating element made of a 2 m long by 1.5 cm diameter tungsten cylinder, emitting 2400 W of power. Assuming the heating element can be considered a perfect blackbody:
(a) What is the wavelength of the peak intensity radiation emitted?
(b) How many photons are emitted per second?

Problem 9:
A beam of radioactive atomic nuclei are moving with a kinetic energy 9 times their rest mass. If the half-life of the atom is 1 second at rest, how long must an observer at rest wait for half of the beam particles to decay? How far do beam particles travel in this time?

Problem 10:
Two uniform infinite sheets of electric charge densities $+\sigma$ and $-\sigma$ intersect at right angles. Choose the coordinate system so that the positively charged sheet lies in the $z = 0$ plane and the negatively charged sheet lies in the $x = 0$ plane.
Find the magnitude and direction of the electric field everywhere and sketch the lines of **E**.

Section II:

Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

Problem 11:

At time t = 0, an already normalized to 1 wave function of an arbitrary potential V(x) is given as $\psi(x, t = 0) = [\psi_1(x) + \psi_2(x)]/\sqrt{2}$, where the $\psi_n(x)$ are orthonormal stationary states of the potential V(x) with respective energies E_n.

(a) Write $\psi(x,t)$, i.e. the wave function now incorporating time.

(b) Find $|\psi(x,t)|^2$. Is it time dependent?

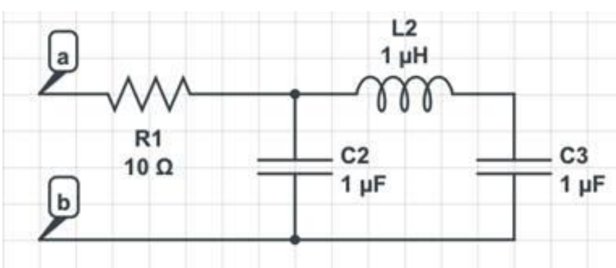
(c) By integration in x and using the orthonormality properties, find the integral of $|\psi(x,t)|^2$ over all space. Is it time dependent? Is this reasonable?

Problem 12:

(a) Calculate the total impedance between nodes (a) and (b) for an AC input $V = V_0e^{i\omega t}$ in symbolic form and using the given values for R_1, L_2, C_2, and C_3?

(b) The current I flowing through R_1 will vary sinusoidally. $I = I_0e^{i(\omega t+\varphi)}$.

What happens to I_0 when ω approaches the resonance frequency? What happens to I_0 when ω is ~ 0.9 or 1.1 times the resonance frequency?



Problem 13:

The radioactive isotope ^{229}Th is an α emitter with a half-life $t_{½\,Th}$ = 7,300 y. Its daughter, ^{225}Ra, is a β emitter with a half-life $t_{½\,Ra}$ = 14.8 d, which is much shorter than the parent's.

(a) Show that the number of ^{229}Th nuclei obey the differential equation

$dN_{Ra}/dt = N_{Th}/\tau_{Th} - N_{Ra}/\tau_{Ra}$,

where N_{Th} is the number of ^{229}Th nuclei and N_{Ra} is the number of ^{225}Ra nuclei,

and τ_{Th} and τ_{Ra} are their mean lives, respectively.

(b) Suppose that at time t = 0 there are N_{Th0} nuclei of the parent isotope ^{229}Th and no nuclei of the daughter isotope ^{225}Ra. Find N_{Th} for t > 0.

(c) Show that after several years, $N_{Ra} = N_{Th}\,\tau_{Ra}/\tau_{Th}$.

Problem 14:

A sphere has a mass m with uniform density ρ and radius a. Calculate the moment of inertia about an axis through its center. Do not just write down the answer, show your calculation.

Problem 15:

(a) Explain in terms of molecular motion why the pressure on the walls of a container increases when a gas is heated at constant volume.

(b) An ideal gas undergoes a process during which the volume V is proportional to $1/P^2$, where P is the pressure. What happens to the temperature T?

(c) A cylinder 2.4 m tall is filled with 0.1 mol of an ideal gas at standard temperature and pressure. The top of the cylinder is then closed with a tight-fitting piston whose mass is 1.4 kg and the piston is allowed to drop until it is in equilibrium.

(i) Find the height of the piston, assuming that the temperature of the gas does not change as it is compressed.

(ii) Suppose that the piston is pushed down below its equilibrium position by a small amount and then released. Assuming that the temperature of the gas remains constant, find the frequency of vibration of the piston.