August 2022 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^8$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Assume that $\psi(x, t)$ is a solution of the Schroedinger equation for a free particle of mass m in one dimension, and that $\psi(x, 0) = A \exp(-x^2/a^2)$.

(a) What is the FWHM in of $|\psi(x, 0)|^2$?

(b) At time t = 0 find the probability amplitude $\phi(p, 0)$ in momentum space.

(c) Find $\psi(x, t)$.

(d) What is the full width at half maximum (FWHM) of $|\psi(x,t)|^2$?

Hint: $\int_{-\infty}^{+\infty} \exp(-a^2(x + c)^2)dx = \sqrt{\pi}/a$

Problem 2:

Consider that a certain distribution of charge and current give rise to the potentials

$$\Phi(\mathbf{r},t) = 0$$

and

$$\mathbf{A}(\mathbf{r},t) = [\mu_0\alpha/(4c)]\,(ct - |x|)^2\,\mathbf{k}, \text{ for } |x| < ct,$$

$$\mathbf{A} = 0 \text{ for } |x| > ct.$$

(a) Find the electric field **E** that results from these potentials and plot it as a function of x.

(b) Find the magnetic field **B** that results from these potentials and plot it as a function of x.

(c) Find the current and charge distributions that produce the potentials given.

Hint: Remember the boundary conditions of electric and magnetic fields.

Problem 3:

Two climbers (let's call them Margaret and Atwood) find themselves in an unfortunate situation. They are tied to one another by a rope of fixed length, but are hanging down from a fixed, frictionless point.

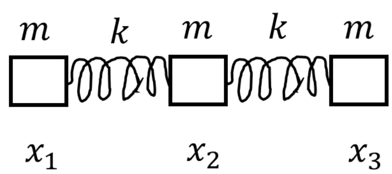
(a) If their ropes stay vertical, find the equations of motion for each climber.

(b) If instead Atwood can swing freely in a plane, find the general equations of motion for the climbers.

Problem 4:

Three identical masses (m) are connected by two identical springs with a spring constant k (as shown below). The masses lie on a frictionless surface and can move just in one dimension.

(a) Write down equations of motion for the coordinates x_1, x_2, and x_3.
(b) Find the angular frequencies of the normal modes and the normal coordinates corresponding to these frequencies.
(c) At time $t = 0$ the mass at x_2 displaced a distance a to the right and the other masses are kept fixed. Then all masses are let go starting from a zero velocity. Determine the time-evolution of the coordinates x_1, x_2, and x_3 thereafter.



Problem 5:
A hydrogen atom is located in a strong magnetic field **B** = B**k**, so that the Zeeman splitting is much larger than the spin-orbit splitting of the energy levels, and to first order the spin-orbit interaction can be ignored. The magnetic moment of the electron is $\boldsymbol{\mu} = (-q_e/(2m_e))(\mathbf{L} + 2\mathbf{S}) = -\mu_B(\mathbf{L} + 2\mathbf{S})/\hbar$.
(a) Find the energies of the 2s and 2p energy levels in the strong magnetic field.

(b) The element neon has a line at 6074 Å. A neon atom is in a 2.5 T field. Assuming that the 6074 Å line comes from a 3p to 3s transition of an exited electron outside of the same core and that you can treat this electron like an electron in hydrogen, what is the energy shift due to the Zeeman effect in eV, if the initial state has m = 1?
(c) What is the change in wavelength due to the Zeeman effect? Give your result both in Angstrom and as a percentage of 6074 Å?

Problem 6:

Assume a photon gas is described by the Planck distribution

$$n(\nu, T) = \frac{8\pi}{c^3}\frac{\nu^2}{\left(e^{\frac{h\nu}{kT}}\right)-1},$$

where ν is the photon frequency and T is the temperature of the gas.

(a) Find the energy density u(T) of the photon gas.

(b) The intensity per unit frequency interval of the radiation emitted by the blackbody is $I(\nu,T) = \frac{1}{4}u(\nu, T)\, c$.

Find an expression for the Stefan-Boltzmann constant σ.

You may find this definite integral useful.

$\int_0^\infty x^n\, dx/(e^x - 1) = n!\, \zeta(n+1)$, where ζ(n) is the Riemann zeta function.

$\zeta(1) = \infty$, $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.202$, $\zeta(4) = \pi^4/90$, $\zeta(5) \approx 1.037$.

Problem 7:

Assume that $\hat{A}$, the operator of an observable A has two eigenstates Φ_1, Φ_2, with eigenvalues a_1, a_2, respectively. Meanwhile, $\hat{B}$, the operator of another observable B has two eigenstates χ_1, χ_2, with eigenvalues b_1, b_2, respectively. The two sets of eigenstates fulfill the following relations

$$\Phi_1 = \frac{2\chi_1+3\chi_2}{\sqrt{13}}, \quad \Phi_2 = \frac{3\chi_1-2\chi_2}{\sqrt{13}}.$$

Both A and B commute with H, but not with each other.

(a) Show that Φ_1, Φ_2, χ_1, and χ_2 have the same energy eigenvalue.

(b) Assume several measurements are made. The first measurement of $\hat{A}$ leads to an eigenvalue of a_1. If we proceed to measure $\hat{B}$ and then measure $\hat{A}$ again, what is the probability of getting a_1 again?

Problem 8:

Consider a grounded conducting sphere of radius R centered at the origin. A point charge q is located on the z-axis at z = 3R/2, and a second point charge -q is located on the z-axis at z = -3R/2.

(a) Find the potential everywhere outside the sphere.

(b) For r >> R, expand the potential in a power series in R/r and keep only terms to up to first order. What is the significance of the terms?