August 2022 Qualifying Exam

Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\lambda)) - 1]^{-1}$

Useful integrals:

$$\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\left(\sqrt{x^2+a^2}\right)}$$

$$\int \sin mx \sin nx \, dx = \frac{\sin(m-n)\,x}{2(m-n)} - \frac{\sin(m+n)\,x}{2(m+n)}, \qquad [m^2 \neq n^2]$$

Section I:

Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

Problem 1:

Use the following masses

$^{56}_{28}Ni$ = 55.942133 u, $^{14}_{4}Be$ = 14.042893 u, neutron = 1.008665 u, and ^{1}H = 1.007825 u

to find

(1) the binding energy of ^{56}Ni,

(2) the binding energy per nucleon of ^{56}Ni,

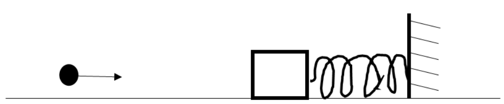
(3) the binding energy of ^{14}Be.

(4) the binding energy per nucleon of ^{14}Be.

1 u = 1.66054×10^{-27} kg = 931.494 MeV/c^2.

Problem 2:

A bullet with mass m is fired horizontally into a wooden block with mass M which lies on a frictionless table (see below). The wooden block is attached to a massless spring that has a spring constant k. The bullet remains embedded in the block after the collision. Based on the maximum displacement d of the wooden block determine the speed of the bullet just before hitting the block.



Problem 3:

Consider a quantum description of a non-relativistic 2D electron gas confined to the x−y plane of a Cartesian coordinate system, with a magnetic field **B** pointed in the z direction. The effect of electronic coupling to the field may be included by modifying the momentum operator **p** = (p_x, p_y) in the Hamiltonian to include a term depending on the vector potential **A** associated with the magnetic field:

$$\mathrm{H} = \frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla} + \mathrm{q_e}\mathrm{A}\right)^2,$$

where q_e is the electron charge. Show that the different (gauge) choices for the vector potential

A = (0, Bx, 0) (Landau gauge),

and

A = ½B(-y, x, 0) (symmetric gauge),

both give the required magnetic field.

Problem 4:

Show that the minimum energy of a simple harmonic oscillator is $\hbar\omega/2$ if $\Delta x \Delta p = \hbar/2$, where $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$.

Problem 5:

The Hamiltonian for an electron interacting with a magnetic field B may be written

$$\mathrm{H} = -\frac{1}{2}\mu\boldsymbol{\sigma}\cdot\mathbf{B} = -\frac{1}{2}\mu\left(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z\right)$$

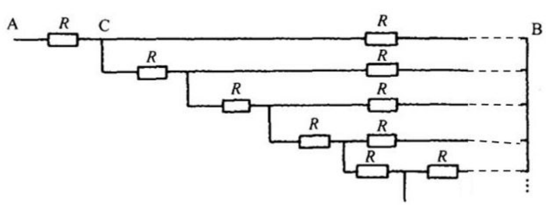
where μ is a constant and the σ_i are Pauli matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Find the eigenvalues corresponding to solutions of H|ψ> = E|ψ> for this system.

Problem 6:

Calculate the equivalent resistance between A and B for this infinite resistor network.



Problem 7:

A frictionless puck is sitting on an inclined plane with gravity pointing down, but there is linear drag of the form −b**v** acting on this puck. The puck has mass m and the incline makes an angle θ with the horizontal.

(a) Find an expression for the terminal velocity of this puck.

(b) Write the equation of motion using this new expression of the terminal velocity.
Define a new variable $u = v - v_{ter}$, to arrive at a simpler differential equation.

(c) Show that a solution of the form $u(t) = Ae^{-Bt}$ can work for this differential equation.
Find the general solution for v(t).

Problem 8:

(a) What kind of plot makes a power law look like a line? How is the slope of that line related to the power law?

(b) Is the mean or the median more sensitive to outliers?

(c) Presuming the statistical uncertainties of an experiment follow a Poisson distribution, by what factor does the statistical uncertainty decrease by, if one obtains four times as much data?

(d) Assume that an experimental quantity is taken from a Gaussian distribution. How does the uncertainty in the mean scale with the number of observations, N?

Problem 9:

A physics professor claims in court that the reason she went through the red light (λ = 650 nm) was that, due to her motion, the red color was Doppler shifted to green (λ = 550 nm). How fast was she going?

Problem 10:

Three charges lie along the x-axis. Charge q_1 = 15 µC is located at x = 2.0 m, and charge q_2 = 6 µC is at the origin.

(a) Where must a negative charge q_3 be placed on the x-axis so that the resultant force on it is zero?

(b) If q_3 is replaced by a negative charge with twice its negative charge, i.e., $q_3' = 2q_3$, would you have to place q_3' in the same position that you found in part (a) if you want that the net force on q_3' is zero? Why?

(c) What would happen if q_3 were positive?

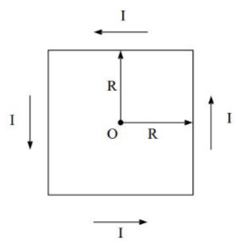
(d) Would it be possible to place q_3 away from the x-axis and still have the resultant force on it to be zero? Why?

Section II:

Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

Problem 11:

(a) Find the magnetic field at the center of a square loop, which carries a steady current I. Let R be the distance from center to side.



(b) Find the field at the center of a regular n-sided polygon, carrying a steady current I. Again, let R be the distance from the center to any side.

(c) Check that your formula reduces to the field at the center of a circular loop in the limit $n \to \infty$.

Problem 12:

A particle of mass m is confined to a one-dimensional region $0 \le x \le a$ by the potential:

$V(x) = 0$ for $0 \le x \le a$,

$V(x) = \infty$ for $x < 0$ or $x > a$.

At $t = 0$ its normalized wave function is $\psi(x, t = 0) = (8/(5a))^{½} [1 + \cos(\pi x/a)] \sin(\pi x/a)$.

(a) What is the wave function at a later time $t = t_0$?

(b) What is the average energy of the system at $t = 0$ and at $t = t_0$?

(c) What is the probability that the particle is found in the left half of the box

i.e., in the region $0 \le x \le a/2$, at $t = t_0$?

Problem 13:

(a) Calculate the center of mass position for a cube of uniform mass density, centered at the origin, whose edges are of length 2 m. (Show this explicitly with a volume integral.)

(b) If we add a point particle with a mass equal to half the total mass of the cube at the position (10, 0, 0) m, find an expression for the CM of the combined cube + particle system.

(c) Now shrink the cube to become a point mass at the origin. If the particle starts moving with a velocity of $\mathbf{v} = (0, v_0, 0)$ m/s, find an expression for the total angular momentum, with respect to the CM of the cube-particle system as a function of time.

Problem 14:

Consider the following matrices:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}.$$

(a) Find S_z following $[S_x, S_y] = i\hbar S_z$.

(b) If $S_{x,y,z}$ describes the spin matrices of a particle, what is the total spin of the particle?

Problem 15:

A very light string of length L_{string} passes over a uniform disk of radius R and mass M. Two masses m_1 and m_2 ($m_1 > m_2$) are attached to the string as shown. The coefficient of static friction between the string and the pulley is large enough, so that the string does not slide over the pulley. If the system starts from the rest, what time does it take for the mass m_1 to move down a distance d? (d is smaller than the distances for the masses to hit the disk).