August 2021 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Consider a tau particle, which is a heavy, unstable relative of the electron with a rest mass of $m_\tau \approx 1.8$ GeV/c^2. A tau can decay intro hadrons (and neutrinos) and has an average lifetime $\tau \approx 0.29$ picoseconds in its rest frame. Suppose that we have a fast-moving tau, generated in the ATLAS detector at the LHC, that is observed to have moved a distance of 2 millimeters inside the detector.

(a) How long would the tau have to live inside the detector for this to be possible?

(b) What is the kinetic energy and momentum of the tau as measured in the detector frame?

(c) Suppose the tau decays to two identical particles of mass m. What is the magnitude of the momentum of each of these particles in the rest frame of the tau?

Problem 2:

(a) Find the energy eigenfunctions and energy levels for a spinless particle confined to a two-dimensional rectangular box, with $|x| < a$ and $|y| < b$.

Do not just write down your answer, but derive it. Justify each step.

(b) Make a diagram or table showing the 5 lowest energy levels and the degeneracies for $b = a$ and for $b = 2a$.

Problem 3:

In the lab frame K, in some volume of space, a constant electric field **E** and a constant magnetic field **B** are present.

Can you find an inertial reference frame K' moving relative to frame K with velocity $\boldsymbol{\beta} = \mathbf{v}/c$, such that the field **E** and **B** are parallel to each other? If yes, what is the velocity $\boldsymbol{\beta}$ of K' relative to K?

Problem 4:

The free energy, F, of a magneto-elastic material is the sum of magneto-elastic and strain energy terms.

(a) Assuming a simple form for the magneto-elastic energy term of $\gamma\varepsilon M^2$, where γ is the magneto-elastic coupling coefficient, ε is the strain and M is the magnetization, write an expression for F in terms of γ, ε, M^2 and Young's modulus, Y.

(b) Assuming $\gamma > 0$, sketch the two contributions to F as a function of ε from negative (compression) to positive (tension), and their sum (= F).

(c) Is the equilibrium value of strain compressive or tensile?

(d) Assuming equilibrium conditions with respect to ε, i.e. $dF/d\varepsilon = 0$, derive an expression for ε in terms of M^2, Y, and γ.

(e) To lowest order in M, does the equilibrium value of strain depend on the sign of M?

Problem 5:

The energy of a single oscillator (measured from its ground state) is given by $E_n = n\hbar\omega$, where $n = 0, 1, 2, \ldots$. Consider a system of N non-interacting oscillators at temperature T.

(a) Find the partition function of this system.

(b) Calculate the average energy of the system and compare with the average energy of a single oscillator.

(c) Calculate the heat capacity of the system and evaluate it in the high temperature limit.

Problem 6:

A conducting ball of radius R_1 carries charges q, and it is concentric with a conducting spherical shell whose inner radius and outer radius are $R_2 = 2R_1$ and $R_3 = 3R_1$, respectively. The shell is grounded. Now a point charge Q is placed at a distance $d = 4R_1$ away from the center of the ball.

(a) Find potential everywhere and the total charge carried by the shell.

(b) The same question as (a) but now the ball and the shell are connected to each other by a thin wire.

Problem 7:

A uniform ball with radius r is rolling down from the top of a fixed sphere with the radius R. Its initial velocity is negligibly small. What will be the angular velocity of the ball when it loses contact with the sphere?

Problem 8:

Consider the one-dimensional harmonic oscillator with the Hamiltonian

$H = P^2/(2m) + \frac{1}{2}m\omega^2X^2$,

with P the momentum operator and X the position operator.

(a) What are the energy eigenvalues E_n of the harmonic oscillator?

(b) Suppose that the harmonic oscillator Hamiltonian is perturbed by $\Delta H = \varepsilon(PX + XP)$.

Calculate the first non-vanishing correction to the ground state energy E_0.

(Hint: Using the creation/annihilation operator method is easiest here.)