August 2020 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

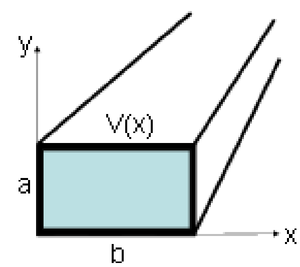
The time-independent Schroedinger equation for an electron of mass m in a spherically symmetric potential V(r) is

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]+\frac{L^2}{2mr^2}+U(r)\right)\psi(r)=E\psi(r),$$

where L is the angular momentum operator. For this problem, assume that the electron has a potential energy function $U(r) = -e^2/r$, where the position of the nucleus is stationary.

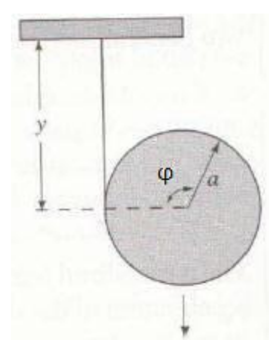
(a) Assume that the electron is in a 1s-like state $\psi(r) \propto \exp(-Br)$. Determine the value of B and the energy E of this state. Express your answers in units of the Bohr radius a_0.
(b) An electron is in the ground state of tritium (^3H), which has one proton and two neutrons. A nuclear reaction instantaneously changes the nucleus to ^3He, which has two protons and one neutron. Calculate the probability that the electron is in the ground state of $^3He^+$ after the transition.
Recall: $\int_0^\infty r^2 e^{-\alpha r}\, dr = 2/\alpha^3$

Problem 2:
A rectangular trough extends infinitely along the z direction, and has a cross section as shown in the figure.



All the faces are grounded, except for the top one, which is held at a potential $V(x) = V_0$. Find the potential inside the trough.

Problem 3:
Consider a uniform disc with mass m and radius *a* that has a massless string wrapped around it with one end attached to a fixed support and allowed to fall with the string unwinding as it falls, as shown in the figure.

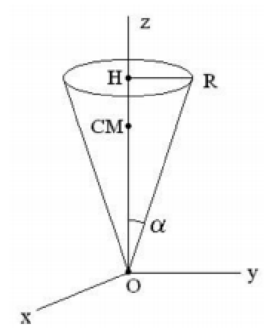


Use y and ϕ as the generalized coordinates to describe the system. Find the equations of motion for y and of the falling disc and the forces of constraint using the method of Lagrange multipliers.

Problem 4:

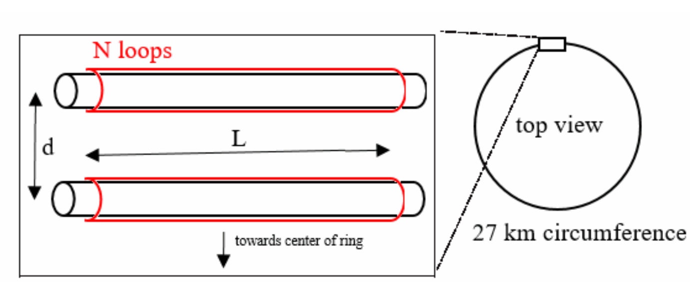
(a) Define all components of the inertia tensor for a continuous mass distribution.

(b) Consider a circular cone of height H and base radius R = H tan α with uniform mass density $\rho = 3M/(\pi H R^2)$ as shown in the figure. Calculate the inertia tensor in the body-fixed coordinate system shown in the figure.



Problem 5:

In the LHC, there are actually two circular tubes, side-by-side, that protons travel through. The protons go one way in one tube, and the other way in the other tube. (And then eventually the two paths are made to cross to smash the protons together.) The diagram below shows a short segment of the two tubes of radius r and separation d (black), with coils of wire on them (red) that carry a current I to generate the magnetic fields. The wires are essentially long straight wires that run right along the sides of the tubes for a length L, and then wrap over the top to the other side. The drawing just shows one red line, but this is supposed to represent N loops of wire forming a bundle with that shape. The wires are coated with an insulator so that current flows along the length of wire only, and not from one wire to another in the bundle.



(a) If we want the protons in the upper tube in the figure to go counterclockwise, and the protons in the lower tube to go clockwise, find the direction of current flow in the upper and lower coils (clockwise or counterclockwise for each).

(b) Near the midpoint of the straight segment of the wire, we can neglect the magnetic fields from the ends, and consider the wires to be just infinite straight, parallel wires. Find the magnetic field (magnitude and direction) at the center of each tube, near the midpoint of the straight sections. Your answer should be in terms of r, d, I, and N.

(c) Given the numbers B = 8 T, r = 50 mm, d = 250 mm, and N = 80, evaluate the current I in the wires.

(d) Ignoring the semicircles at the ends, calculate the force (magnitude and direction) on the bundle of wires on one side of a pipe due to the bundle of wires on the other side of the pipe. First find a symbolic equation in terms of N, I, L, and r, then plug in the numbers above and L = 15 m.

Problem 6:

At t = 0 an electron in the hydrogen atom occupies the combined position and spin state

$$\sqrt{\frac{1}{3}}\psi_{210}(r,\theta,\phi)\chi_{+} + \sqrt{\frac{2}{3}}\psi_{211}(r,\theta,\phi)\chi_{-}\ .$$

(a) If you measure E, what value(s) might you get, and with what probability(ies)?
(b) If you measure L^2, what value(s) might you get, and with what probability(ies)?
(c) If you measure L_z, what value(s) might you get, and with what probability(ies)?
(d) If you measure S^2, what value(s) might you get, and with what probability(ies)?
(e) If you measure S_z, what value(s) might you get, and with what probability(ies)?
(f) What is the probability density for finding the electron at r, θ, ϕ in terms $\psi_{nlm}(r,\theta,\phi)$.
(g) What is the probability per unit length for finding the particle with spin up a distance r from the proton in terms of $R_{nl}(r)$?

Problem 7:

During phase transition, the change in the pressure P and temperature T can be expressed by the Clausius-Clapeyron relation,

$dP/dT = L/(T\ \Delta V)$,

where L is the latent heat (per kg) and ΔV is the change in volume (per kg).
(a) How much pressure does one have to put on an ice cube to make it melt at -1° C?
The density of ice is 917 kg/m^3, and the latent heat of 1 kg of ice melting is 333000 J.
(b) Approximately how deep under a glacier does it have to be before the weight of the ice above gives the pressure you found in part (a)?

Problem 8:

Consider a particle of charge q and mass m in the presence of a constant, uniform magnetic field $B = B_0\,\mathbf{k}$, and of a uniform electric field of amplitude E_0, rotating with frequency ω in the (x,y) plane, either in the clockwise or in counterclockwise direction.
Let $\mathbf{E} = E_0\cos(\omega t)\,\mathbf{i} \pm E_0\sin(\omega t)\,\mathbf{j}$.

(a) Write down the equation of motion for the particle and solve for the Cartesian velocity components $v_i(t)$ in terms of B_0, E_0, and ω if $\omega \neq \omega_c = qB_0/m$ (the cyclotron frequency).
Show that, if $\omega = \omega_c$, a resonance is observed for the appropriate sign of ω.
Hint: Let $\zeta = v_x + iv_y$, and solve for ζ(t).
(b) Solve for the Cartesian velocity components $v_i(t)$ at resonance.
(c) Now assume the presence of a frictional force $\mathbf{f} = -m\gamma\mathbf{v}$, where $\mathbf{v}$ is the velocity of the particle. Find the general solution for ζ(t) for clockwise rotation of the electric field, and find the steady state solution (t >> 0) when $\omega = \omega_c$.